CONSUMER AND PRODUCER SURPLUS OF THE LINEAR DEMAND AND SUPPLY FUNCTIONS BY FUZZIFYING POLYGONAL FUZZY NUMBERS AND DEFUZZIFYING GRADED MEAN

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Abstract

Let $p = a - bx$ and $p = e + gx$ be linear demand function and linear supply function, respectively. If the price p is fixed then the quantity x is not necessarily fixed. In this paper, the quantity x are fuzzified using polygonal fuzzy number. Then the graded mean method is used to defuzzify the fuzzy demand function and fuzzy supply function. Finally the results obtained show that our polygonal fuzzy model gives more optimal results than triangular fuzzy model.

Keywords: *Graded mean, polygonal fuzzy number, consumer surplus, producer surplus*

1. INTRODUCTION

In recent years, fuzzy set theory has become a main tool to study economic problems such as estimating optimal revenue [3,4] and optimal profit [7,11], calculating the best prices of two and three mutually complementary merchandises [13,14] and calculating the consumer and producer surplus [10,12].

In [12], Yao and Wu considered linear demand function and linear supply function in which the demand quantity and the supply quantity are triangular fuzzy numbers. Then he calculated the consumer surplus and producer surplus. Following paper, Wu [10] estimated these surpluses taking into consideration demand function and supply function to be linear and quadratic. He fuzzify the coefficients instead of quantity. In the both of the papers [10,12], triangular fuzzy numbers has been used for fuzzification.

In this paper, we use the linear demand $p = a - \mathrm{b}x$ of function and linear supply $p = e + \mathrm{g}x$ of function to calculate the consumer surplus and producer surplus. Then we fuzzify the quantity x by using polygonal fuzzy number. Yao and Wu [10,12] has used the centroid method for defuzzification. Here we use the graded mean defuzzification method defined in [5] and formulated in [9]. Finally, we showed that our polygonal fuzzy model gives more optimum results than Yao and Wu's [12] triangular fuzzy model.

2. PRELIMINARIES

A fuzzy number is a function X from $\mathbb{R}$ to $[0,1]$, satisfying:

1) X is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$;
2) X is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\lambda \in [0,1], X(\lambda x + (1-\lambda)y) \geq \min\{X(x), X(y)\}$;
3) X is upper semi-continuous;
4) the closure of $\{x \in \mathbb{R} : X(x) > 0\}$, denoted by X^0, is compact [2].

We denote the set of all fuzzy numbers by $F(\mathbb{R})$. Note that the function a_1 defined by

$$a_1(x) := \begin{cases} 1 & , \quad \text{if } x = a, \\ 0 & , \quad \text{otherwise,} \end{cases}$$

where $a \in \mathbb{R}$, is a fuzzy point [6].

Definition 2.1[5]. *Graded Mean of a fuzzy number* A *defined as*

$$G(A) = \frac{1}{2}\frac{\int_0^1 \alpha\,[A_L(\alpha) + A_R(\alpha)]\,d\alpha}{\int_0^1 \alpha},$$

where $A_L(\alpha)$ *is the left* $\alpha - cut$ *and* $A_R(\alpha)$ *is the right* $\alpha - cut$ *of the fuzzy number* A .

There are two identical definitions of the polygonal fuzzy number in the literature. Here we will use the notation of Baez-Sanchez et al.'s [1].

Definition 2.2[1]. *For a fixed partition*

$$P_m :\ 0 = \alpha_0 < \alpha_1 < \dots < \alpha_m = 1$$

of the interval $[0,1]$, $A \in F(\mathbb{R}^n)$ *is called a polygonal fuzzy set associated with partition* P_m . $\forall \alpha \in (0,1]$ *we have that the* $\alpha -$ *level set* A_α *satisfies*

$$A_\alpha = (1-\lambda)\,A_{\alpha_i} + \lambda A_{\alpha_{i+1}}$$

where $0 \le \alpha_i < \alpha < \alpha_{i+1} \le 1$ *for some* $i = 0,\dots,m-1$ *and* $\lambda = \lambda(\alpha) = (\alpha - \alpha_i)/(\alpha_{i+1} - \alpha_i)$.

We denote the set of all polygonal fuzzy numbers by $F_P(\mathbb{R})$. It is clear that $F_P(\mathbb{R}) \subset F(\mathbb{R})$. Now let us briefly review the operations of summation and scalar multiplication on the set $F_P(\mathbb{R})$. Suppose $A = (x_1, x_2, x_3, \dots, x_n) \in F_P(\mathbb{R})$, $B = (y_1, y_2, y_3, \dots, y_n) \in F_P(\mathbb{R})$ and $\alpha \in \mathbb{R}$ then we have

$$A + B = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

$$\alpha \cdot A = \begin{cases} (\alpha x_1, \alpha x_2, \alpha x_3, \dots, \alpha x_n) & \alpha \ge 0 \\ (\alpha x_n, \alpha x_{n-1}, \alpha x_{n-2}, \dots, \alpha x_1) & \alpha < 0 \end{cases} \text{[8]}.$$

For a fixed partition $0 \le \alpha \le 1$ and $n \in Z^+$ the graded mean of $(2n+1) -$ polygonal fuzzy number $A = (x_1, x_2, x_3, \dots, x_{2n+1})$ formulated as:

$$G(A) = \frac{1}{6n^2}\begin{pmatrix} x_1 + 6x_2 + \dots + 6(n-2)\,x_{n-1} + 6(n-1)\,x_n + \\ (6n-2)\,x_{n+1} + 6(n-1)\,x_{n+2} + \dots + 6x_{2n} + x_{2n+1} \end{pmatrix} \text{[9]}.$$

3. RESULTS

We consider the demand function

$$p = a - \mathrm{b}x \quad \left(0 \le x \le \frac{a}{b}\right),$$

and the supply function

$$p = e + \mathrm{g}x \quad (x \ge 0),$$

where $a > e > 0$, $b > 0$, $e > 0$ and $g > 0$ are fixed numbers and p is the unit price with respect to quantity x. In the real situation, for a fixed price p, the demand quantity x will have small changes. We characterize this changes using polygonal fuzzy numbers. Our techniques is similar to that of [12].

For a fixed unit price p we fuzzify the quantity x to a polygonal fuzzy number as

$$D = (x - \varepsilon_1, x - \varepsilon_2, \dots, x - \varepsilon_n, x, x + \varepsilon_{n+1}, \dots, x + \varepsilon_{2n-1}, x + \varepsilon_{2n}) \tag{3.1}$$

where $0 < \varepsilon_n < \varepsilon_{n-1} < \dots < \varepsilon_2 < \varepsilon_1$ and $x < \varepsilon_{n+1} < \varepsilon_{n+2} < \dots < \varepsilon_{2n-1} < \varepsilon_{2n}$. Similarly, for a fixed price p, the supply quantity x is not necessarily fixed as x $\left(\frac{a-p}{b}\right)$ in reality. For a fixed unit price p, we fuzzify the quantity x to a polygonal fuzzy number as

$$S = (x - \delta_1, x - \delta_2, \dots, x - \delta_n, x, x + \delta_{n+1}, \dots, x + \delta_{2n-1}, x + \delta_{2n}) \tag{3.2}$$

where $0 < \delta_n < \delta_{n-1} < \ldots < \delta_2 < \delta_1$ and $x < \delta_{n+1} < \delta_{n+2} < \ldots < \delta_{2n-1} < \delta_{2n}$. The graded means of the polygonal fuzzy numbers D and S are

$$G(D) = E_d(x) = x + \frac{1}{6n^2}(-\varepsilon_1 - 6\varepsilon_2 - \ldots + 6\varepsilon_{2n-1} + \varepsilon_{2n})$$

and

$$G(S) = E_s(x) = x + \frac{1}{6n^2}(-\delta_1 - 6\delta_2 - \ldots + 6\delta_{2n-1} + \delta_{2n})$$

respectively, where $E_d(x)$ estimates the demand quantity in the fuzzy sense when the price is $p \quad (= a - \mathrm{b}x)$, and $E_s(x)$ estimates the supply quantity in the fuzzy sense when the price is $p \quad (= e + \mathrm{g}x)$.

Now we calculate the fuzzy demand function and fuzzy supply function using the polygonal fuzzy numbers defined in the Equations (3.1) and (3.2). We have the fuzzy demand function

$$P = a_1 - b \cdot D$$

which is called fuzzy price for demand. Then we have

$$P = (a - b(x + \varepsilon_1), \ldots, a - b(x + \varepsilon_n), a - \mathrm{b}x, a - b(x - \varepsilon_{n+1}), \ldots, a - b(x - \varepsilon_{2n})) \qquad (3.3)$$

and its graded mean can be easily calculated as

$$G(P) = E_1(x) = a - b\left(x + \frac{1}{6n^2}(-\varepsilon_1 - 6\varepsilon_2 - \ldots + 6\varepsilon_{2n-1} + \varepsilon_{2n})\right). \qquad (3.4)$$

Similarly we have the fuzzy supply function $P = e_1 + g \cdot S$ which is called fuzzy price for supply. Then we have

$$P = (e + g(x - \delta_1), \ldots, e + g(x - \delta_n), e + \mathrm{g}x, e + g(x + \delta_{n+1}), \ldots, e + g(x + \delta_{2n})), \qquad (3.5)$$

and its graded mean can be easily calculated as

$$G(P) = E_2(x) = e + g\left(x + \frac{1}{6n^2}(-\delta_1 - 6\delta_2 - \ldots + 6\delta_{2n-1} + \delta_{2n})\right) \qquad (3.6)$$

where $E_1(x)$ and $E_2(x)$ are polygonal fuzzy demand and polygonal fuzzy supply function, respectively.

Now we recall the crisp consumer surplus and crisp producer surplus for the linear case as follows. Let x_* be equilibrium quantity. Then the consumer surplus for the crisp case is

$$\mathrm{CSC} = \frac{1}{2}\mathrm{b}x_*^2$$

and the producer surplus for the crisp case is

$$\mathrm{PSC} = \frac{1}{2}\mathrm{g}x_*^2.$$

Now we are ready to give the formulizations of consumer surplus and of producer surplus using the polygonal fuzzy numbers.

Proposition 3.1. *If the demand equation is*

$$P = a_1 - b \cdot D,$$

and the supply function is

$$P = e_1 + g \cdot S$$

then the consumer surplus for the fuzzy case is

$$\text{CSF} = \frac{1}{2} b \left(\frac{1}{b+g} \begin{bmatrix} a - e - \frac{1}{6n^2} b(-\varepsilon_1 - 6\varepsilon_2 - \dots + 6\varepsilon_{2n-1} + \varepsilon_{2n}) - \\ \frac{1}{6n^2} g(-\delta_1 - 6\delta_2 - \dots + 6\delta_{2n-1} + \delta_{2n}) \end{bmatrix} \right)^2,$$

and the producer surplus for the fuzzy case is

$$\text{PSF} = \frac{1}{2} g \left(\frac{1}{b+g} \begin{bmatrix} a - e - \frac{1}{6n^2} b(-\varepsilon_1 - 6\varepsilon_2 - \dots + 6\varepsilon_{2n-1} + \varepsilon_{2n}) - \\ \frac{1}{6n^2} g(-\delta_1 - 6\delta_2 - \dots + 6\delta_{2n-1} + \delta_{2n}) \end{bmatrix} \right)^2$$

where the coefficients defined in Equations (3.3) and (3.5).

Proof. First we should find the equilibrium quantity. We set $E_1(x)$ equal to $E_2(x)$ and solve:

$$a - b\left(x + \frac{1}{6n^2}(-\varepsilon_1 - 6\varepsilon_2 - \dots + 6\varepsilon_{2n-1} + \varepsilon_{2n})\right)$$
$$= e + g\left(x + \frac{1}{6n^2}(-\delta_1 - 6\delta_2 - \dots + 6\delta_{2n-1} + \delta_{2n})\right).$$

Hence we have

$$x = x_* = \frac{1}{b+g} \begin{bmatrix} a - e - \frac{1}{6n^2} b(-\varepsilon_1 - 6\varepsilon_2 - \dots + 6\varepsilon_{2n-1} + \varepsilon_{2n}) - \\ \frac{1}{6n^2} g(-\delta_1 - 6\delta_2 - \dots + 6\delta_{2n-1} + \delta_{2n}) \end{bmatrix}$$

by using the Equations (3.4) and (3.6). The equilibrium price can be determined by using $E_1(x)$ or $E_2(x)$ as:

$$p_* = E_1(x_*) = a - \mathrm{b}x_* - \frac{1}{6n^2} b(-\varepsilon_1 - 6\varepsilon_2 - \dots + 6\varepsilon_{2n-1} + \varepsilon_{2n}),$$
$$p_* = E_2(x_*) = e + \mathrm{g}x_* + \frac{1}{6n^2} g(-\delta_1 - 6\delta_2 - \dots + 6\delta_{2n-1} + \delta_{2n}).$$

Then we can calculate the consumer surplus as

$$\int_0^{x_*} \left[a - b\left(x + \frac{1}{6n^2}(-\varepsilon_1 - 6\varepsilon_2 - \dots + 6\varepsilon_{2n-1} + \varepsilon_{2n})\right) - p_* \right] \mathrm{d}x = \frac{1}{2}\mathrm{b}x_*^2.$$

Similarly, we get the producer surplus as

$$\int_0^{x_*} \left[p_* - \left(e + g\left(x + \frac{1}{6n^2}(-\delta_1 - 6\delta_2 - \dots + 6\delta_{2n-1} + \delta_{2n})\right)\right)\right] \mathrm{d}\,x = \frac{1}{2}\mathrm{g}x_*^2.$$

4. DISCUSSION

We use [12, Example 2.1] in order to compare our results and Yao and Wu's results [12]. Let the demand function be

$$p = 22 - 2x, 0 \le x \le 11$$

and supply function be

$$p = 10 + 2x, x \geq 0$$

where our coefficients are $a = 22$, $b = 2$, $e = 10$ and $g = 2$.

We set

$$22 - 2x = 10 + 2x.$$

Then equilibrium quantity is $x_* = 3$ and equilibrium price is $p_* = 16$. Hence we have the consumer surplus and producer surplus:

$$\mathrm{CSC} = \frac{1}{2}\mathrm{b}x_*^2 = \frac{1}{2} \cdot 2 \cdot 3^2 = 9,$$
$$\mathrm{PSC} = \frac{1}{2}\mathrm{g}x_*^2 = \frac{1}{2} \cdot 2 \cdot 3^2 = 9.$$

If we take

$$\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_{2n} \text{ and } \delta_1 = \delta_2 = \dots = \delta_{2n},$$

then fuzzy sense is just crisp sense. This shows that our model is a generalization of the crisp model.

We use 5-polygonal fuzzy number. Let $\varepsilon_4 - \varepsilon_1 = 1$, $\delta_4 - \delta_1 = -1$, $2\varepsilon_3 - 2\varepsilon_2 = 0$ and $6\delta_3 - 6\delta_2 = 0$. Then the equilibrium quantity is

$$x_* = \frac{1}{b+g}\left[a - e - \frac{1}{24}b(\varepsilon_4 + 6\varepsilon_3 - 6\varepsilon_2 - \varepsilon_1) - \frac{1}{24}g(\delta_4 + 6\delta_3 - 6\delta_2 - \delta_1)\right]$$

$$= \frac{1}{4}\left[12 - \frac{1}{24}2(\varepsilon_4 + 6\varepsilon_3 - 6\varepsilon_2 - \varepsilon_1) - \frac{1}{24}2(\delta_4 + 6\delta_3 - 6\delta_2 - \delta_1)\right].$$

If we choose

$$\varepsilon_4 - \varepsilon_1 = 1 , \ \delta_4 - \delta_1 = -1, \quad 6\varepsilon_3 - 6\varepsilon_2 = 0 \text{ and } 6\delta_3 - 6\delta_2 = 0,$$

then the equilibrium quantity is

$$x_* = 3.$$

The equilibrium price p_* can be calculated as

$$p_* = E_1(x_*) = a - \mathrm{b}x_* - \frac{1}{6n^2}b(\varepsilon_4 + 6\varepsilon_3 - 6\varepsilon_2 - \varepsilon_1) = 15.917.$$

Hence consumer surplus and producer surplus estimate as

$$\mathrm{CSF} = \frac{1}{2}\mathrm{b}x_*^2 = \frac{1}{2} \cdot 2 \cdot 3^2 = 9$$
$$\mathrm{PSF} = \frac{1}{2}\mathrm{g}x_*^2 = \frac{1}{2} \cdot 2 \cdot 3^2 = 9.$$

Let

$$\varepsilon_4 - \varepsilon_1 = 1 , \ \delta_4 - \delta_1 = 1 , \ 6\varepsilon_3 - 6\varepsilon_2 = 0 , \ 6\delta_3 - 6\delta_2 = 0 .$$

Then equilibrium quantity x_* is

$$x_* = 2.9583$$

and the equilibrium price p_* is

$$p_* = E_1(x_*) = 22 - 2 \cdot x_* - \frac{1}{24} \cdot 2 \cdot (\varepsilon_4 + 6\varepsilon_3 - 6\varepsilon_2 - \varepsilon_1) = 16.$$

Hence consumer surplus can be calculated as

$$\text{CSF} = \frac{1}{2} \text{b} x_*^2 = 8.7515.$$

The total savings to consumers who are willing to pay a higher price for the product is 8.7515.

Similarly, the producer surplus can be calculated as

$$\text{PSF} = \frac{1}{2} \text{g} x_*^2 = 8.7515$$

The total gain to producers who are willing to supply units at a lower price is 8.7515.

Let $\varepsilon_4 - \varepsilon_1 = -1, 6\varepsilon_3 - 6\varepsilon_2 = -10.8,\ \ 6\delta_3 - 6\delta_2 = -10.8$ and $\delta_4 - \delta_1 = -1$.

We have the equilibrium quantity as

$$x_* = 3.45$$

and the equilibrium price p_* can be calculated as

$$p_* = E_1(x_*) = 22 - 2 \cdot x_* - \frac{1}{24} \cdot 2 \cdot (\varepsilon_4 + 6\varepsilon_3 - 6\varepsilon_2 - \varepsilon_1) = 16.$$

Consequently, we have

$$\text{CSF} = \frac{1}{2} \text{b} x_*^2 = 11.903$$

$$\text{PSF} = \frac{1}{2} \text{g} x_*^2 = 11.903.$$

Now we sum the above results in Table 1.

Case	x_* [12]	x_* Polygonal case	p_* [12]	p_* Polygonal case	PSF/CSF [12]	PSF/CSF Polygonal case
1 (Crisp)	3	3	16	16	9	9
2 (fuzzy)	3	3	15.333	15.917	9	9
3 (fuzzy)	2.667	2.9583	15.999	16	7.113	8.7515
4 (fuzzy)	3.333	3.45	16.001	16	11.109	11.903

Table 1. CSF/PSF For the polygonal fuzzy numbers

We note that if we require that the coefficients $\varepsilon_4 - \varepsilon_1$ and $\delta_4 - \delta_1$ to be compatible with Yao and Wu's [12] triangular fuzzy numbers, they can not be chosen arbitrarily.

We argue that these PSF/CSF values can be improved by choosing more adaptive polygonal fuzzy numbers.

Case	ε_1/δ_1	ε_2/δ_2	ε_3/δ_3	ε_4/δ_4	x_*	p_*	PSF/CSF
1	2	1	0.75	1	3.0625	16	9.3789
2	2	1.50	0.25	1	3.3125	16	10.973
3	2	1.9	0.1	1	3.45	16	11.903
4	2	1.99	0.01	1	3.495	16	12.215

Table 2. The coefficients $\varepsilon_2, \varepsilon_3, \delta_2$ and δ_3 has been changed under the conditions $\varepsilon_4 - \varepsilon_1 = -1$ and $\delta_4 - \delta_1 = -1$.

Table 1 and Table 2 show that polygonal fuzzy model gives more optimum results than not only crisp model but also triangular fuzzy model.

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