THE OPTIMAL SUPPLY OF CONGESTED PUBLIC GOODS

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Abstract
Impure public goods resulting from the congestion effect are discussed in the literature solely for the case of homogenous populations where consumers have identical demands. We extend this to include heterogeneous populations, where demands are rectangularly distributed. We compare the optimal values of the control variables (quantity of the public good and the number of users) for both homogeneous and heterogeneous populations, as well as the social optimum values for both cases.

Key words: *Congested Public Good, Lindhal, Rectangulary Distributed Demand, Homogeneous and Heterogeneous Customers*

1. INTRODUCTION

The issue of congested public goods that maintain on the one hand the non-exclusion principle and on the other hand also maintain some degree of rivalry has been discussed in the economic literature over the last few decades. Sandler (2002) mentions some impure public goods that include ocean fisheries, pest-control, curbing organized crime, and eliminating acid rain. All these items display rivalry but remain non-excludable.

The congested public goods case can be thought of as an extension of the Club theory developed by Buchanan (1965) in his seminal paper that attempted to bridge the gap between a pure private good, which is excludable and rivalrous in consumption, and a pure public good, that holds for goods whose consumption is non- rivalrous and non-excludable. The club goods are excludable and subject to some rivalry in the form of congestion, while the congested public good can either be excludable or not, although rivalry definitely exists due to congestion or crowding effects.

The original model of Buchanan assumes a club good being consumed by a homogeneous population whose individuals possess the same tastes and endowments. In the following years more extended models were developed and scenarios of heterogeneous populations were also discussed. For example, Sandler (1977) used the assumption of a distribution of net willingness to pay, while Fraser and Hollander (1992) assume homogeneous populations in tastes but with different individual incomes.

Many more theoretical and empirical papers dealing with congestion (or crowding) effects for club and public goods can be found in a survey named "Club Theory: Thirty Years Later" of Sandler and Tschirhart "(1997).

Still, the congested public good is a topic that continues to interest economists up till this very day. For example, the recent paper of Birulin (2006) also discusses the issue of public goods with congestion where the capacity (termed G) is given, while the number of consumers (users), *N*, is determined (which means that some other consumers are excluded for the sake of ex-post efficient allocation).

It turns out, some what surprisingly, that the goods of larger capacity, i.e. those which exhibit less exclusion, may be easier to finance than goods of smaller capacity (see pp. 290 third paragraph)

Recently Vianroux (2008) introduced in a short note the assumption of unobserved heterogeneity in the marginal utility of income of a given size population dealing with its effect on the estimation of urban transportation demand with endogenous traffic congestion.

We conclude from the above that these combined issues of two optimal control values from the social point of view, i.e. the optimal quantity of the congested public good, G, and the optimal number of users, *N*, should be investigated.

Another question that should also be considered is that of to what degree may the heterogeneity of a population affect the demand for the congested public good, *G*, on the one hand and the optimal number of users, *N*, on the other hand. Would the homogeneous population desire more public goods and services due to their homogeneity in taste and income, or would they desire less public goods and services? Moreover, is the optimal number of the homogeneous population which has identical demands larger or smaller than the optimal number of the non-homogeneous population which is endowed with different tastes, income, and other socio-economic characteristics. A policy maker who recognizes his habitants should ask himself this kind of question since the supply of different kinds of public goods (either pure or congested) consumes a major share of the public budget.

Furthermore, is the optimal population size under a regime of congested public goods of the homogeneous population larger or smaller than in the case of a heterogeneous population? It can be applied to the club goods discussed by Buchanan when rivalry of some degree exists may lead to a very similar analysis, i.e. should we prefer a club with a homogeneous or heterogeneous population? If a population is diversified in socio-economic characteristics, should we supply more of the congested public good to a larger number of members in such a club or vice versa?

We compare the social optimum values of *G* and *N* of both cases in order to gain insight into which case the congestion effect is more crucial and may either reduce or increase the optimal number of users, N, on the one hand and decrease or increase the demand for the congested public good, *G*, on the other hand.

We use a very simple framework where linear demand curves are vertically summed up in Lindhal style where all customers have identical demands, D_i , for *G*, and where the demands are distributed rectangularly around the average customer whose demand is D_i, similar to the case above where all customers have the same demand, D_i . An interesting question that arises is: what is the level of tolerance of different populations towards congestion?.

Some comparative static analysis is included in our paper to examine the influence of congestion effects (i.e. negative externalities) on both the quantity demanded of congested public goods and the number of users.

In the next section we develop both cases and make comparisons between different kinds of populations along with appropriate numerical examples. The final section consists of implications and conclusions.

2. THE MODEL

In this section we develop two cases where the number of users, *N*, and the optimal value of the congested public good, *G*, is derived and compared for homogeneous and non-homogeneous populations.

In the first case below we compare those solutions where the congestion effect reduces the demand and the consumer surplus thereby reducing the social welfare of the society. In case 2 the congestion effect does not affect the demand, but increases the production costs of the congested public good, and by that it also reduces the social welfare.

Case 1: Homogeneous Population with Identical Demands

We define in equation (1.1) below the demand of each of *N* customers where index *i* represents the demand of a representative consumer *i* :

(1.1) $p_i = A_i - \alpha \cdot G - \beta \cdot N$

The demand is linear and A_i represents the reservation price of each consumer, *i*, α is the slope of the demand curve and β is a parameter that measures the (negative) sensitivity of price towards congestion. By vertical summation we can write equation (1.2):

(1.2) $P = \sum_{i=1}^{N} p_i = N \cdot p_i = N \cdot A_i - \alpha \cdot N \cdot G - \beta \cdot N^2$

In addition we assume a simple linear cost function of producing and supplying the public good *G*

(1.3) $TC = c \cdot G$

where *c* represents the cost of each unit of the public good (average or marginal).

In a simple framework of a Lindhal solution where the vertical summation of demand is equal to the social marginal cost we get (1.4):

(1.4) $P = \sum_{i=1}^{N} p_i = N \cdot p_i = c$

Since $\sum_{i=1}^{N} p_i = c$, i.e., *N* customers cover the cost *c* of each unit of the public good, we get (1.5) as follows:

(1.5) $N \cdot A_i - \alpha \cdot N \cdot G - \beta \cdot N^2 = c$

Because we assume *N* identical (homogeneous) customers we can assume that:

A_i=*A*, i.e., identical reservation prices for all *N* customers.

Therefore all *N* customers share the cost burden equally and each pays C/N.

From (1.5) we derive the optimal value of public good, *G**:

(1.6) $G^* = \frac{A - (c/N) - \beta \cdot N}{\alpha}$

Thus, the consumer surplus of each consumer is:

(1.6') $CS_i = \frac{\left[A - \beta \cdot N - (c/N)\right]^2}{2\alpha}$

For this optimal solution we calculate the optimal number, *N*, of public good users that will "allow" the maximum total summation of consumer surplus, *TCS*, as follows:

(1.7) $TCS = \sum_{i=1}^{N} CS = \frac{\left(A \cdot N - c - \beta \cdot N^2\right)^2}{2\alpha \cdot N}$

By taking the derivative of *TCS* with respect to *N* at (1.8):

$$(1.8)\quad \frac{\partial TCS}{\partial N} = \frac{2\cdot\left(A\cdot N - c - \beta\cdot N^2\right)\cdot\left(A - 2\beta\cdot N\right)\cdot 2\alpha\cdot N - 2\alpha\cdot\left(A\cdot N - c - \beta\cdot N^2\right)^2}{4\alpha^2\cdot N^2} = 0$$

we get the optimal number of users, N^*:

$$(1.9)\quad N^* = \frac{A + \sqrt{A^2 + 12\beta\cdot c}}{6\beta}$$

and from (1.6) and (1.9) we find the optimal value of G in terms of all relevant parameters as follows:

$$(1.10)\quad G^* = \frac{A}{1.5\alpha} - \frac{12\beta\cdot c}{1.5\alpha\cdot\left(A + \sqrt{A^2 + 12\beta\cdot c}\right)}$$

If we define the demand curve (see equation (1.1)) as net price over cost, c, i.e., $(P_i - c)$ then we can rewrite (1.9) and (1.10) as (1.9') and (1.10'):

$$(1.9')\quad N^* = \frac{A}{3\beta}$$

$$(1.10')\quad G^* = \frac{A}{1.5\alpha}$$

and the optimal TCS is:

$$(1.11)\quad TCS^* = \frac{2A^3}{27\alpha\beta}$$

A simple comparative static analysis with respect to A and β leads to very simple and intuitive conclusions that are summarized in (1.12), (1.13) and (1.14).

$$(1.12)\quad \frac{\partial G}{\partial A} > 0,\ \frac{\partial G}{\partial \beta} = 0,$$

$$(1.13)\quad \frac{\partial N}{\partial A} > 0,\ \frac{\partial N}{\partial \beta} < 0,$$

$$(1.14)\quad \frac{\partial TCS}{\partial A} > 0,\ \frac{\partial TCS}{\partial \beta} < 0,$$

The congestion effect does not affect the optimal value of G but reduces the number of users since policy makers wish to avoid the congestion effects. By so doing they minimize the total consumer surplus losses.

Case 2: Heterogeneous Population with a Rectangulary Distributed Demand

In this case customer *i* has the demand curve:

$$(1.15)\quad D_i:\ p_i = A + N/2 - i - G - \beta\cdot N$$

For the median customer $i = \frac{N}{2}$ in the rectangularly (uniformly) distributed demand we face the same demand as (1.1) above of a representative consumer where all customers are identical. Using a vertical summation once again and assuming that p_i is the net cost of c, leads to the following "vertical" demand for the public good:

$$(1.15') \quad P = \begin{cases} A \cdot N + \frac{N^2}{2} - \frac{(1+N)\cdot N}{2} - \beta \cdot N^2 - N \cdot G & G \le A + N/2 - N - \beta \cdot N \\ \frac{(N-i)\cdot(N-i-1)}{2} & G > A + N/2 - N - \beta \cdot N \end{cases}$$

Based on Appendix A the optimal value of $\tilde{N}$ in the case of heterogeneous consumers is as follows:

$$(1.16) \quad \tilde{N} = \frac{\left(\beta A - \beta/2 + 1/2\right) + \sqrt{\left(\beta A - \beta/2 + 1/2\right)^2 - 3\cdot\left(\beta^2/2 - 1/24\right)\cdot\left(\frac{A^2 - A}{2} + 13/12\right)}}{3\cdot\left(\beta^2/2 - 1/24\right)}$$

and therefore the optimal value of the congested public good of a heterogeneous population at equilibrium, $\tilde{G}$, is as follows:

$$(1.17) \quad \tilde{G} = A - 1 - \left(\beta - 1/2\right)\cdot\tilde{N}$$

The main comparative static analysis results are summarized below:

$\frac{\partial \tilde{G}}{\partial A} > 0$ if $\beta < 1/2$, but the sign $\frac{\partial \tilde{G}}{\partial A}$ is ambiguous if $\beta > 1/2$

$\frac{\partial \tilde{G}}{\partial \beta} > 0$, if $\beta < 1/2$, and $\frac{\partial \tilde{G}}{\partial \beta} < 0$ if $\beta > 1/2$

Still, $\frac{\partial \tilde{N}}{\partial \beta} > 0$ and $\frac{\partial \tilde{N}}{\partial A} > 0$,

Using the optimal values $\tilde{N}$ and $\tilde{G}$ of scenario 2 we now compare this to the similar values of scenario 1 above.

First we compare the N^* *and* $\tilde{N}$ values that we derived at (1.9') and (1.16)

$$(1.18) \quad \tilde{N} = \frac{\left(\beta A - \beta/2 + 1/2\right) + \sqrt{\left(\beta A - \beta/2 + 1/2\right)^2 - 3\cdot\left(\beta^2/2 - 1/24\right)\cdot\left(\frac{A^2 - A}{2} + 13/12\right)}}{3\cdot\left(\beta^2/2 - 1/24\right)} \gtrless \frac{A}{3\beta} = N$$

The second comparison is with respect to the $\tilde{G}$ and G^*

(1.19) $\tilde{G} = A - 1 - \left(\beta - 1/2\right) \cdot \tilde{N} \underset{<}{\overset{>}{}} A - \beta \cdot N = G$

The third comparison is with respect to $T\tilde{C}S$ *and TCS*

(1.20) $T\tilde{C}S = \left(\beta^2/2 - 1/24\right) \cdot \tilde{N}^3 - \left(\beta A - \beta/2 + 1/2\right) \cdot \tilde{N}^2 + \left(\frac{A^2 - A}{2} + 13/12\right) \cdot \tilde{N} \underset{<}{\overset{>}{}} \frac{2A^3}{27\beta} = TCS^*$

Three different regions are considered.

From (1.18), (1.19) and (1.20) above we can derive the relationship between the optimal values of *G*, *N*, and *TCS* for homogeneous and non-homogeneous populations.

a. If $0 < \beta < 1/\sqrt{12}$ then $\tilde{N} < N^*$ and $\tilde{G} < G^*$

b. If $1/\sqrt{12} < \beta < 1/2$ then $\tilde{N} > N^*$ and $\tilde{G} > G^*$

For significantly small values of the congestion coefficient, β, the optimal number of non-homogeneous customers is smaller and they benefit from smaller amounts of the congested public good then under the parallel terms in the case of a homogeneous population. The picture dramatically changes in terms of optimal population size and the optimal quantity of the congested public good for somewhat larger values of coefficient β.

c. However, when $\beta > 1/2$ then we get $\tilde{N} > N$ and the relationship between $\tilde{G}$ and *G* is ambiguous

For β larger than 1/2 we find that at optimum more heterogeneous consumers should be benefiting from the supply of the congested public good. On the other hand, the optimal quantity supplied of a congested public good, G, for the homogeneous case can be either larger or smaller than for the non-homogeneous case. As a result the comparison between TCS and $T\tilde{C}S$ is unclear and requires further investigation.

Since we cannot solve it analytically, what remains is to solve it by the use of simulations[1] as introduced in Table 1 below.

Table 1: Comparison between Optimal Variables for Different A and β Parameters

	Small A	**Medium A**	**High A**
Small β	$N^* > \tilde{N}$ $G^* > \tilde{G}$ $TCS^* > T\tilde{C}S$	$N^* > \tilde{N}$ $G^* > \tilde{G}$ $TCS > T\tilde{C}S$	$N^* > \tilde{N}$ $G^* > \tilde{G}$ $TCS^* > T\tilde{C}S$
Medium β	$N^* < \tilde{N}$ $G^* > \tilde{G}$	$N^* < \tilde{N}$ $G^* < \tilde{G}$	$N^* < \tilde{N}$ $G^* < \tilde{G}$

[1] Available on request

	$TCS^* > T\tilde{C}S$	$TCS^* > T\tilde{C}S$	$TCS^* > T\tilde{C}S$
High β	$N^* < \tilde{N}$ $G^* > \tilde{G}$ $TCS^* \underset{<}{\overset{>}{-}} T\tilde{C}S$	$N^* < \tilde{N}$ $G^* > \tilde{G}$ $TCS^* \underset{<}{\overset{>}{-}} T\tilde{C}S$	$N^* < \tilde{N}$ $G^* \underset{<}{\overset{>}{-}} \tilde{G}$ $TCS^* > T\tilde{C}S$

With no congestion effect, or only a very small congestion effect the existence of a homogeneous society (population) is socially preferable, since it leads to a higher quality and quantity of the public good and a larger use by the community who enjoys it, thus, a higher level of social welfare is achieved. However, this picture may change in the case of a larger congestion effect. The most interesting results are that the congestion affects the optimal values of *G* and *N* and that they move in opposite directions.

3. CONCLUSIONS

In the model discussed above we determined the optimal values of public goods and number of users in cases where the population is either homogeneous or heterogeneous under congestion that negatively affects the demand for the public good. The size of the public good and the number of users may be different and can be affected not only by congestion but also by the degree of necessity of the public good (represented by A), and as a result also affects the total social welfare derived from the production and use of the congested public good. We summarize below the main important conclusions.

A policy maker who recognizes his population (whether they are relatively homogenous in their income and taste and other socioeconomic characteristics or the degree of diversity reflects heterogeneity) should take this into account in the decision making process. The decisions should address the following issues:

a. What is the optimal size of each population group that is served by any kind of public good?

b. If the public good is not pure but congested how does that affect the size of the population group?

When there are no congestion affects at all, the optimal number of public good users as well as the optimal quantity of the pure public good used are both higher with homogeneous populations than with heterogeneous populations.

A homogeneous population desires a higher level of public good services in both quality and quantity terms that are to be shared by a larger optimal size of the population compared to the scenario where the population is heterogeneous in income and utility terms.

As a result the total consumer surplus (or the social welfare) is also higher in the former case.

Congestion negatively affects the optimal number of public good users in the case of a homogeneous population. Therefore the TCS of the homogeneous population declines with higher values of Congestion (high ß). This differs sharply from the case of a heterogeneous population where a higher Congestion leads to an increase in the optimal value of congested public good use. Thus, TCS may increase since the Congestion effect can be contradictory on heterogeneous populations. The total effect of Congestion on the total consumer surplus of a heterogeneous population is ambiguous. Based on the above analysis we can conclude that for higher CD the optimal number of homogeneous users (N^*) declines while that of the heterogeneous users ($\tilde{N}$) is increasing, such that at some point $N^* < \tilde{N}$. On the other hand the comparison of the parallel values of optimal G^* and $\tilde{G}$ can be different. G is never affected either by congestion generated by demand, however, demand congestion

affects the optimal $\tilde{G}$ value either negatively or positively. Thus, at some point when Congestion is small $G^* > \tilde{G}$, however the sign is subject to switching i.e. $G < \tilde{G}$ especially in the case of a high demand for a public good (a high A). As a result, the relationships between TCS and $\tilde{TCS}$ are also ambiguous.

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