THE DISPROPORTION OF ALLOCATION UNDER THE GIVEN BOUNDARY CONDITIONS

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Abstract

There are several well known measures of the inequality of distribution of goods and burdens. One of them is for example the Gini coefficient. Applicability of such measures is limited when the use of disproportionate allocation was intended. This is the case in the allocation of seats in the European Parliament. The Treaty of Lisbon provides that the distribution is done according to degressive proportionality with the specified minimum and maximum number of seats allocated to the smallest and largest Member State. These two conditions called boundary conditions at the outset determine the deviation from the proportional division. Therefore, there arises a problem of determining the measure of the disproportion of allocation under the given boundary conditions. This article comprises the proposals for such measures and their application for the analysis of the distribution of seats in the European Parliament.

Keywords: *degressive proportionality, measure of degression, the boundary conditions, Gini coefficient*

1. INTRODUCTION

The widely-approved for centuries in European culture principle of allocation of goods and burdens is derived from Aristotle's principle, according to which, each unit participating in the distribution receives a share proportional to its value among all other units. The value of units may be understood differently. This may be its importance, property, or as in the case of the electoral matters - the population of the respective constituency. In this situation, this means that, for example, seats in the collegial bodies should be allocated in proportion to population of the districts between which the seats are distributed. The first considerations relating to proportionality in the electoral problems date back to the turn of the eighteenth and nineteenth centuries. The issue was presented in the context of distribution of seats in the House of Representatives of the US Congress and mathematical formalization made mainly in the twentieth century.

If the sequence $0 < p_1 \leq p_2 \leq \ldots \leq p_n$ represents the population of each constituency or generally the value of units involved in the operation, then this positive sequence $q_1, q_2, \ldots, q_n$ is proportional with respect to $0 < p_1 \leq p_2 \leq \ldots \leq p_n$ if and only if $p_1/q_1 = p_2/q_2 = \ldots = p_n/q_n$. The numbers $q_1, q_2, ..., q_n$ are generally real numbers and the sum $q_1 + q_2 + ... + q_n$ is the total number of shared goods. This sequence is called a sequence of quotas of proportional distribution and with a fixed amount of distributed goods it is determined unequivocally.

In many cases the values of the distribution must be integers. All allocations of seats in the collegial bodies are integers by their nature. This means that you cannot always assume a sequence of quotas as a solution to the problem of allocation. In such cases, one most often seeks an appropriate approximation of sequences with integer values. In practice, there were several of such approximations. Their names derive from the surnames of the researchers who proposed the solutions, i.e. methods of Hamilton, Jefferson, Webster, Adams and Hill (Young 2003). Obviously, this does not exhaust the list of all the proposals but merely historically indicates the most important ones based on the universality of application or precision of mathematical work.

The principle of proportional allocation works well with small discrepancies in the population of the constituencies or generally small differences between the individual units involved in the allocation. Otherwise in the case of fixed, limited amount of goods it can lead to marginalization of certain

groups. Particularly, when allocating seats in collegial bodies, some units with a small population may be deprived of their share altogether. Their share expressed by a real number may be so small that with certain types of rounding will not constitute a single seat. Such a situation often does not gain social acceptance. It is not acceptable in particular by the European community during the process of establishing representations of Member States for the European Parliament. It turns out that the differences in the population of Member States, particularly the smallest and largest state of Malta and Germany are so large that because of the desire to ensure the proper representation for each country the proportional principle of Aristotle has to be abandoned. Thus it came to define of the rule of degressively proportional distribution of wealth and burdens.

2. DEGRESSIVE PROPORTIONALITY

The principle of degressively proportional allocation of goods and burdens has been legally sanctioned in the Treaty of Lisbon. It was introduced as a rule for distribution of seats in the European Parliament. The exact wording of the relevant provision (The Treaty of Lisbon, article 9A) is as follows:

"*The European Parliament shall be composed of representatives of the Union's citizens. They shall not exceed seven hundred and fifty in number, plus the President. Representation of citizens shall be degressively proportional, with a minimum threshold of six members per Member State. No Member State shall be allocated more than ninety-six seats*".

The treaty, however, does not contain any explanation of what the legislator meant by degressively proportional distribution, and only in the resolution "Proposal to amend the Treaty provisions concerning the composition of the European Parliament" (Lamassoure & Severin 2007) one can find a suitable clarification. It has been included in two statements: "the larger the population of a country, the greater its entitlement to a large number of seats" and "the larger the population of a country, the more inhabitants are represented by each of its Members of the European Parliament". With this additional information, you can make an attempt to mathematicaly formalize the idea indicated in the Treaty of Lisbon.

Definition: Positive sequence $s_1, s_2, \ldots, s_n$ is degressively proportional with respect to $0 < p_1 \leq p_2 \leq \ldots \leq p_n$ if and only if $s_1 \leq s_2 \leq \ldots \leq s_n$ and $p_1/s_1 \leq p_2/s_2 \leq \ldots \leq p_n/s_n$.

The sequence $s_1, s_2, ..., s_n$ in this case is a sequence of elements denoting the number of total seats allocated, while the sequence $p_1, p_2, ..., p_n$ is a sequence, whose values represent the population of individual Member States of the European Union. Such allocation is called in the literature (Delgado-Marquez et al., 2013) "rounded degressive proportionality" (RDP). Sometimes in practical applications, for example in the so-called Cambridge Compromise, the requirement to meet by sequence $s_1, s_2, ..., s_n$ the condition $p_1/s_1 \leq p_2/s_2 \leq ... \leq p_n/s_n$ is not required. It is then obligatory that the condition is fulfilled only before rounding to integers. So understood degressive proportionality is called in the literature (Delgado-Márquez et al. 2013) "unrouded degressive proportionality" (UDP). Its legal acceptance is included in a report (Gualtieri & Trzaskowski 2013), where a corresponding European Parliament resolution draft was placed, which stipulates that „the ratio between the population and the number of seats of each Member State before rounding to whole numbers shall vary in relation to their respective populations in such a way that each Member of the European Parliament from a more populous Member State represents more citizens than each Member from a less populous Member State and, conversely, that the larger the population of a Member State, the greater its entitlement to a large number of seats".

Regardless of which version of degressively proportional distribution will be considered, that is to say whether it would be required to meet the relevant inequality before or after rounding, possible solutions are many (Łyko & Rudek 2013). Therefore, a problem arises with the choice of a particular allocation, but also with the possible assessment of decisions taken. This is all the more difficult because in official documents there can be found only one tip for this issue. There is a provision in Alain Lamassoure' and Adrian Severin's resolution (Lamassoure, Severin 2007), according to which

the minimum and maximum numbers set by the Treaty „must be fully utilised to ensure that the allocation of seats in the European Parliament reflects as closely as possible the range of populations of the Member States". Taking this into account as a criterion for the assessment of individual solutions one must accept their highest compatibility with proportional distribution. Therefore, it is reasonable to apply for this purpose the indicators used to assess the degree of disproportionalities in distribution.

3. MEASURES OF DISPROPORTIONALITY OF DISTRIBUTION

In the literature one can find many disproportionality measures used in the social and economic analyses. There are several measure groups distinguished on the basis of their specificity. For example, in his work (Karpov 2008) divides disproportionate measures into absolute value indices (maximum deviation index, Rae index, Loosmore-Hanby index, Grofman index and Lijphart index) and quadratic indices (Gallagher index, Monroe index, Gatev index, Rybatsev index, and Szalai index). To investigate the question of disproportionality one also uses measures of inequality. The most common of them, functioning in the literature for over a hundred years is the Gini coefficient. Measures of inequality may in matters described herein be used in several ways. For example, the Gini coefficient can measure the degree of inequality in population / seats ratio as related to the allocation of seats in the collegial bodies. It can also be a measure of inequality of the quotient seats / population. Most often the results are different in these two cases. But the Gini coefficient can also be calculated by treating seats as a kind of goods that anyone who is interested will receive. For example, in the case of distribution of seats in the EP between Member States it can be considered that the amount of "good" that a resident receives of the state with the population p_i is equal to the quotient s_i/p_i, where s_i is the number of seats that this state has. In the following discussion, the Gini coefficient inequality will be understood as a vector with coordinates s_i/p_i. Out of the measures of disproportionality five of them will be used for analysis. Table 1 provides basic information about these measures (notation: $s_i^* = \frac{s_i}{\sum_{i=1}^{n} s_i}$, $p_i^* = \frac{p_i}{\sum_{i=1}^{n} p_i}$).

Table 1. Chosen measures of disproportions

Index	Formula	Remarks
Rae	$I_{Rae} = \frac{1}{n}\sum_{i=1}^{n}\left\|s_i^* - p_i^*\right\|$	Value depends on the number of contenders. The upper limit is $2/n$
Gallagher	$L_{sq} = \sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(\left\|s_i^* - p_i^*\right\|\right)^2}$	Least squares coefficcient
Gini	$G = \frac{\sum_{i,j=1}^{n}\left\|x_i - x_j\right\|}{2n\sum_{i=1}^{n} x_i}$	$x_i = \frac{s_i}{p_i}$
Saint Lague	$SL = \sum_{i=1}^{n} p_i^*\left(\frac{s_i^*}{p_i^*} - 1\right)^2$	No upper limit
Szalai	$I_{\text{Szalai}} = \sqrt{\frac{\sum_{i=1}^{n}\left(\frac{s_i^* - p_i^*}{s_i^* + p_i^*}\right)^2}{n}}$	Sensitive to changes in small states' representations

Source: own studies.

The literature also contemplates the disproportionate class of coefficients based on the angle formed by vectors $p = (p_1, p_2, .., p_n)$ and $s = (s_1, s_2, .., s_n)$. The properties of these measures are similar to feature of Gallagher's measures (Koppel & Diskin 2009).

4. THE RELATIVE MEASURES OF THE DISPROPORTIONATE DISTRIBUTION

Applying general measures of distribution disproportionality, particularly those regulated ones, does not give us proper assessment of the issue. It is known that in the case of allocation of seats to the European Parliament it is not possible to use the entire possible set of values of individual indicators. In general, if we assume that an allocation has to additionally meet specific boundary conditions, it limits the variation range of the individual indicators. The Lisbon Treaty specifies such boundary conditions. Namely it indicates the minimum number of seats available for the smallest state and the maximum for the biggest state. These restrictions, namely 6 and 96 are given in the form of inequality but currently the interpretations tend to accept them as the exact number of seats allocated to Malta and Germany. The term boundary condition is often understood as the overall number of shared goods, which in our example is equal to the total number of seats, i.e. 751 (Łyko 2013) and (Dniestrzański 2014).

Table 2. Values of disproportionality ratios for the distribution of seats in the EP with the use of divisor methods and distribution in the term 2014-2019

A	B	C	D	E	F
		Seats			
Member State	Population	Adams	Jefferson	Webster	2014-2019
Germany	81 843 743	119	123	121	96
France	65 397 912	95	98	96	74
United Kingdom	62 989 550	92	94	93	73
Italy	60 820 764	89	91	90	73
Spain	46 196 276	68	69	68	54
Poland	38 538 447	56	57	57	51
Romania	21 355 849	32	32	32	32
Netherlands	16 730 348	25	25	25	26
Greece	11 290 935	17	16	17	21
Belgium	11 041 266	17	16	16	21
Portugal	10 541 840	16	15	16	21
Czech Republic	10 505 445	16	15	16	21
Hungary	9 957 731	15	14	15	21
Sweden	9 482 855	14	14	14	19
Austria	8 443 018	13	12	12	19
Bulgaria	7 327 224	11	11	11	17
Denmark	5 580 516	9	8	8	13
Slovakia	5 404 322	8	8	8	13

Finland	5 401 267	8	8	8	13
Ireland	4 582 769	7	6	7	11
Croatia	4 398 150	7	6	6	11
Lithuania	3 007 758	5	4	4	11
Latvia	2 055 496	3	3	3	8
Slovenia	2 041 763	3	3	3	8
Estonia	1 339 662	2	2	2	6
Cyprus	862 011	2	1	1	6
Luxembourg	524 853	1	0	1	6
Malta	416 110	1	0	1	6
Total	**508 077 880**	**751**	**751**	**751**	**751**
Rae		0,0007	0,0007	0,0003	0,010
Gallagher		0,0034	0,034	0,0015	0,0481
Gini		0,0567	0,0950	0,0492	0,4147
Saint-Lague		0,0015	0,0025	0,0007	0,2505
Szalai		0,0683	0,2690	0,0578	0,3327

Source: own studies.

You will notice that the rounding of sequence of quotas causes the indicators of the allocation disproportionality measures to produce results different than zero. For the analysis of indicators and the population of the Member States in January 2012 the results are presented in Table 2. There is therefore no distribution, for which disproportionality ratios are equal to zero. The mere rounding quotas to the integers causes that these values are positive. This exception to the strict proportionality causes slight variations but provides information to what extent rounding affects the growth of disproportionality ratios.

Disproportionality measurement values given in Table 2 are in vast majority close to zero. The exception is the high value of Szalai for the distribution using Jefferson's method. Jefferson's Method is a divider method with rounding down to the nearest integer. This is a method most unfavorable for small contenders. At the same time Szalai index is very sensitive to distribution mismatch with regards to the smaller contenders.

The boundary conditions cause the deviation from proportionality to be significantly greater than that which would be an effect of rounding only. It is justified therefore to ask how much the deviation of the disproportionate rates from zero is forced by the imposed boundary conditions. The answer to this question is important as in order to have a distribution closest to proportional as possible one needs to know within what limits of coefficient values one can operate. If, for example, the Gini coefficient for the distribution of seats in the term 2014-2019 is 0.4147 (column F Table 2), without the knowledge of what is its lower limit one cannot state how far the distribution is from possible proportionality.

Figure 1 illustrates how the Lorenz curve deviates from proportionality when the proportion disorder is a result of introduction of a lower limit of the quotas of goods that must be supplied to each contestant. The figure is illustrative and does not reflect true values for the population of the Member States of the European Union. Doubled field area between the blue lines (proportional distribution) and dotted red and black is equal to the lower limiting factor.

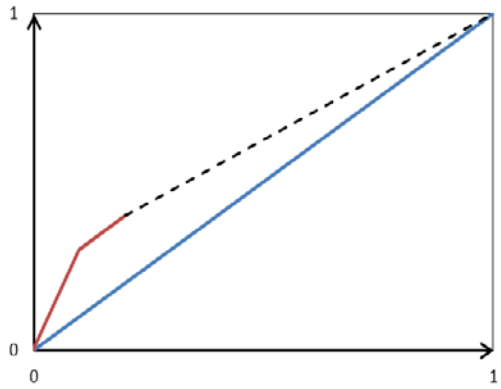


Fig. 1. Illustration to determine the lower limit of the Gini coefficient for the distribution of seats in the EP

Source: own work.

Taking advantage of the fact that the 7 smallest Member States in the EU has quotas lower than 6 and must have at least 6 seats in the European Parliament one can determine the lower limit value of the Gini coefficient. It amounts to 0.0361. There is therefore no degressively proportional distribution meeting the boundary conditions for which the Gini coefficient is less than 0.0361.

Thus set the lower limit of the Gini coefficient is only an approximation of its actual minimum value taken on a class of degressively proportional distributions with established boundary conditions. In order to know this value one would have to determine the value of such coefficient for all possible distributions and select the smallest number form among those. Using the corresponding algorithm included in the work (Łyko, Rudek 2013) one can solve this task. Table 3 shows the minimum and maximum values for the selected disproportionality coefficients for degressively proportional distributions with the boundary conditions established in the Treaty of Lisbon, determined on the basis of the population of Member States in January 2012.

Table 3. Minimum and maximum values for the selected disproportionality coefficients for degressively proportional distribution with the boundary conditions established in the Treaty of Lisbon

Member State	**RAE**		**GALLAGHER**		**GINI**		**SAINT LAGUE**		**SZALAI**	
	min	max	min	max	min	max	min	max	min	max
Germany	96	96	96	96	96	96	96	96	96	96
France	87	77	87	77	77	77	87	77	87	77
Great Britain	84	75	85	75	75	75	85	75	85	75
Italy	84	73	83	73	73	73	83	73	83	73
Spain	64	56	64	56	56	56	64	56	64	56
Poland	54	47	54	47	56	47	54	47	54	47
Romania	30	27	30	27	37	27	30	27	30	27
Netherlands	24	22	24	22	29	22	24	22	24	22
Greece	17	15	17	15	20	15	17	15	17	15
Belgium	17	15	17	15	20	15	17	15	17	15

Portugal	17	15	17	15	20	15	17	15	17	15
Czech Rep.	17	15	17	15	20	15	17	15	17	15
Hungary	17	15	17	15	19	15	17	15	17	15
Sweden	17	15	17	15	19	15	17	15	17	15
Austria	16	15	16	15	17	15	16	15	16	15
Bulgaria	14	15	14	15	15	15	14	15	14	15
Danmark	11	15	11	15	12	15	11	15	11	15
Slovakia	11	15	11	15	12	15	11	15	11	15
Finland	11	15	11	15	12	15	11	15	11	15
Ireland	10	15	10	15	11	15	10	15	10	15
Croatia	10	15	10	15	11	15	10	15	10	15
Lithuania	7	15	7	15	8	15	7	15	7	15
Latvia	6	15	6	15	6	15	6	15	6	15
Slowenia	6	15	6	15	6	15	6	15	6	15
Estonia	6	15	6	15	6	14	6	14	6	14
Cyprus	6	10	6	10	6	11	6	11	6	11
Luksembourg	6	7	6	7	6	7	6	7	6	7
Malta	6	6	6	6	6	6	6	6	6	6
	751	751	751	751	751	751	751	751	751	751
	0.0056	0.0108	0.0308	0.0509	0.3898	0.4896	0.01771	0.5675	0.2943	0.3947

Source: own work.

Knowing the smallest and the largest value of the considered coefficients for the distributions of the said class one can construct relative measures of distribution disproportionality. It is enough to transform them all using the formula $I_R(P) = \frac{I(P) - I_{\min}}{I_{\max} - I_{\min}}$, where $I(P)$ is here value of disproportionality coefficient I for the distribution P, $I_{\min}$ and $I_{\max}$, and the minimum and maximum value of the coefficient I for given boundary conditions and thus obtain indicators that for all concerned degressively proportional distributions take the values ranging from zero to one. This allows for a more accurate assessment of individual solutions. As the distributions that are impossible to implement due to the principle of digressive proportionality or the imposed boundary conditions are not taken into account. Additionally, there is also a possibility of drawing comparisons of solutions using different measures, which is not possible without prior normalization. Of course, if the measure calculated in such way is closer to zero, the distribution is closer to proportional representation. Indications that are close to unity suggest a large deviation from the proportional distribution.

The current distribution of seats for the European Parliament does not satisfy the condition of degressive proportionality but the values of all the analyzed factors are set within the ranges set for degressively proportional allocation. Therefore, the disproportionality of distribution can be assessed with the use of relative measures calculated for the analyzed indices. Comparison of the results is presented in Table 4.

Table 4. Values of relative disproportionality measures the for the current distribution of seats in the European Parliament

RAE	GALLAGHER	SAINT LAGUE	SZALAI	GINI
0,8462	0,8607	0,4236	0,3825	0,2495

Source: own work.

As one can notice the results do not allow for an unequivocal assessment. The spread of relative measures is large as from 0.2495 for the Gini coefficient to 0.8607 in the case of Gallagher index. This shows the difference in the aspects of disproportionality highlighted by each measure. The lesson here is that this type of analysis should be conducted using the largest possible number of indicators.

5. SUMMARY

Despite the inclusion in the EU resolution the record, that the allocation of seats in the European Parliament should reflect as closely as possible the range of Populations of the Member States there are no indicators, on the basis of which one could estimate the measure of distribution disproportionality. Commonly used measures do not take into account deviations from proportionality that is forced by introduction of boundary conditions of the distribution, and secondly because of the different range of values it is not possible to compare results obtained with different methods. These inconveniences are eliminated by consideration of relative measures proposed in this article. With the ability to determine a set of all degressively proportional distributions that meet certain boundary conditions, one can specify the range of variation in the indicators and thus give their relative values. An example of implementation to the current distribution of seats in the European Parliament shows that depending on the choice of measure the conclusions may be different. Moreover, the use of five indicators already causes significant discrepancies. This reinforces the demand to clearly define in legal documents a verification method of the proposed solutions with respect to their incompatibility with proportional distribution.

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REFERENCES

Ceriani, L., Verme, P. (2014). Individual Diversity and the Gini Decomposition. Social Indicators Research. Vol 121, Issue 3, 637-646.

Dniestrzański, P., (2014). *Proposal for measure of degressive proportionality*. Procedia – Social and Behavioral Sciences, 110, 140–147.

Delgado-Márquez, B., Kaeding, M.,Palomares, A., (2013). *A more balanced composition of the European Parliament with degressive proportionality.* European Union Politics. September 2013 14: 458-471.

Grimmett, G., Laslier, J. F., Pukelsheim, F., Ramirez–González, V., Rose, R., Słomczyński, W., Zachariasen, M., &Życzkowski, K. (2011). *The allocation between the UE Member States of seats in the European Parliament*. Studies PE 432.760.

Gualtieri, R., & Trzaskowski, R. (2013). *European Parliament resolution of 13 March 2013 on the composition of the European Parliament with a view to the 2014 elections*. (2012/2309(INI)).

Hill, J. (1911), *Letter to William C. Huston, Chairman, House Committee on the Census, dated April 25, 1911* (in:) U.S. Congress, House, Apportionment of Representatives, House Report 12, 62d Congress, 1st Session.

Huntington E.V. (1921), *The Mathematical Theory of the Apportionment of Representatives*. Proceedings of the National Academy of Sciences, USA, t. 7, 123-127.

Karpov, A. (2008). *Measurement of disproportionality in proportional representation system*. Mathematical and Computer Modelling 48, 1421-1438

Koppel, M., Diskin, A. (2009). *Measuring disproportionality, volatility and malapportionment: axiomatization and solution*. Soc Choice Welf, 33: 281-286.

Lamassoure, A., Severin, A. (2007). European Parliament Resolution on „Proposal to amend the Treaty provisions concerning the composition of the European Parliament" adopted on 11 October 2007 (INI/2007/2169).

Łyko, J., (2013). *Boundary, Internal and Global Measure of the Degression of Distribution*. Procedia Economics and Finance, 5, 519-523.

Łyko, J., Rudek, R. (2013). *A fast exact algorithm for the allocation of seats for the EU Parliament*. Expert Systems with Applications, 40 (13), 5284–5291. http://dx.doi.org/10.1016/j.eswa.2013.03.035.

Pukelsheim, F., (2007). *A Parliament of Degressive Representativeness*? Institut für Mathematik, Universität Augsburg, Preprint Nr 015/2007.

Pukelsheim, F., (2010). Putting citizens first: Representation and power in the European Union. In: M. Cichocki and K. Życzkowski, K., (Eds.) Institutional Design and Voiting Power In European Union. London: Ashgate, 235-253.

Rose, R., Bernhagen, P., Borz, G., (2012). *Evaluating competing criteria for allocating parliamentary seats*. Mathematical Social Sciences 63 (2),85–89.

Saint-Lague A. (1910), *La Representation et la methode des moindrescarres*. ComptesRendus de l'Academie des Sciences, t. 151, 377-378.

Young, H. P. (2003). *Sprawiedliwy podział*. WN SCHOLAR. Warszawa.

The Treaty of Lisbon. http://europa.eu/lisbon_treaty/ full_text/index_en.htm [Accessed:15.02.2012].