ON THE RISK MODELING IN GENERAL INSURANCE

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Abstract

In the present work are considered some approaches for risk modeling in general insurance. Some problems are described which are not subject of the classical risk theory and their reformulation in the actuarial practice are made. Numerical examples to determine the amount of insurance premiums in order to limit to one percent the probability of insolvency are presented. It is shown that depending on the claims size distribution, the Normal approximation and Normal power transformation may lead to differences in the estimation of the safety loadings.

Key words: *statistical distribution, models, insurance, claim, risk*

INTRODUCTION

The occurrence of damage is a prerequisite for a claim. If the claim is larger than the estimated insurance premium, the insurer could be in a losing position. The loss size may adopt different values depending on the circumstances, which have stochastic nature. In the theory, the insurance risk is considered like the standard deviation of the amount of loss distribution, defined as a random variable.

To be possible the modelling the insurance risk, the following conditions must be fulfilled (Hart 1996):

- the loss should be considered as a random variable, i.e. may adopt random values, which in the initial moment are unknown;
- the circumstances around the loss should be subject to definition;
- should not have too much exposure to risk;
- the premium should be consistent with the risk estimation and the insurance market;

CLASSICAL RISK THEORY

To reduce the risk for insolvency for a given period, the insurer may take some of the following methods (Hart 1996):

1. Increasing og the insurance capital.
2. Increasing of the individual profit.
3. Limiting the maximum permitted level of risk.
4. Limiting of the individual risks (with reinsurance).
5. Increasing the number of risks (above given level, which depends on the individual profits).
6. Limiting the correlation between the risks.

MODELING THE RISK OF INSOLVENCY

Let's U be the initial reserve of the insurance portfolio, and N is a Poisson distributed random variable with expectation n; $X_i, i = 1,\ldots,N$ - the individual claims sizes, which are independent, identically

distributed (i.i.d.) random variables with mean $E\{X_i\} = m$ and second moment about the origin $\alpha_2 = E\{X_i^2\}$; P - the premium size, that was received at the start of the year.

The total size of the claims is expressed as the sum of random number of i.i.d. random variables

$$C = X_1 + \ldots + X_N$$

The mathematical expectation of this random variable is

$$E\{C\} = nm$$

The number of the claims has Poisson distribution and consequently, its mean is equal to the variance. Thus, for the variance of the distribution of the total number of claims we obtain

$$var\{C\} = E\{N\}var\{X_i\} + var\{N\}[E\{X_i\}]^2 = nE\{X_i^2\} = n\alpha_2$$

The risk premium of that portfolio is the mean $E\{C\} = nm,$ and the received risk premium can be expressed as

$$P = (1+\lambda)nm,$$

where $\lambda > 0$ are the necessary safety loadings. In practice, to ensure return of the shareholders, these reserves usually are expressed as a percent of the net premium size and are designed for winning stocks. Thus the price, which is available for the claims is

$$U + (1+\lambda)nm$$

and the insurer may be in a situation of bankruptcy in the end of the year if

$$U + (1+\lambda)nm - C \leq 0$$

On the other hand, it is not reasonable to accumulate too large reserves. The basic problem is to minimize the probability of insolvency, i.e.

$$Pr\{U + (1+\lambda)nm - C < 0\} \leq \varepsilon,$$

where ε is the maximum permissible probability for insolvency. This can be transformed in the following way

$$Pr\{C > U + (1+\lambda)nm\} \le \varepsilon$$

and after standardizing we obtain

$$Pr\{\frac{C - nm}{\sqrt{n\alpha_2}} > \frac{U + \lambda nm}{\sqrt{n\alpha_2}}\} \le \varepsilon \quad (1)$$

MODELLING OF THE FREE RESERVES

Founding values for U, which satisfy inequality (1), we obtain information about the size of the free reserve, that the insurer should provide, depending on the maximum permissible probability for insolvency ε. If the portfolio is large enough, for claim size could be used the Normal approximation (Raeva, Pavlov 2015) and the Central Limit Theorem leads that the distribution of C may be approximated by the Normal distribution, and

$$\frac{C - nm}{\sqrt{n\alpha_2}} \approx N(0,1)$$

is approximately standard distributed random variable.

If we choose z_ε (using the results of the tables for Standard Normal distribution), such that $\Phi(z_\varepsilon) = 1 - \varepsilon$, then (1) may be written as

$$\frac{U + \lambda nm}{\sqrt{n\alpha_2}} \ge z_\varepsilon ,$$

where

$$U \ge z_\varepsilon \sqrt{n\alpha_2} - \lambda nm \quad (2)$$

If we use Normal power transformation (Raeva, Pavlov 2015) for the claims size, from the first order of Cornish-Fisher expansion (Fisher, Cornish 1960) we have

$$x_\varepsilon = z_\varepsilon + \frac{\gamma(z_\varepsilon^2 - 1)}{6},$$

where γ is the coefficient of asymmetry of the considered distribution. Then, for the free reserve we obtain

$$U = (z_{\varepsilon} + \frac{\gamma(z_{\varepsilon}^2 - 1)}{6})\sqrt{n\alpha_2} - \lambda nm$$

(3)

From (2) and (3) we can obtain the necessary safety reserves, which the insurer should include in the premium to limit the probability for insolvency to given level ε.

For example, if we consider an insurer's portfolio with annual mean of the number of claims $n = 100$ and free reserves $U = 100\ 000$ €, and the distribution of the claims size presented in Table 1.

Table 1 - Claim size distribution

Number of intervals	Claims size intervals	Mean of the intervals	Claims frequency in the interval (%)
	€	X_i	p_i
1	0 - 100	79	0,05
2	100 - 250	194	0,83
3	250 - 500	390	3,88
4	500 - 1 000	759	11,99
5	1 000 - 2 500	1 696	32,27
6	2 500 - 5 000	3 571	26,16
7	5 000 - 10 000	6 930	16,52
8	10 000 - 25 000	14 448	7,3
9	25 000 - 50 000	32 644	0,88
10	50 000 - 100 000	63 587	0,11
11	over 100 000	130 151	0,01

For the first three moments about the origin we have

$$m = \sum_{i=1}^{11} X_i \frac{p_i}{100} = 4159{,}03$$

$$\alpha_2 = \sum_{i=1}^{11} X_i^2 \frac{p_i}{100} = 4{,}3 \times 10^7$$

$$\alpha_3 = \sum_{i=1}^{11} X_i^3 \frac{p_i}{100} = 1{,}1 \times 10^{12}.$$

A well known fact is that, in the common case, the number of claims follows Poisson distribution (Hossack, Pollard, Zenhwirth 1983). Accepting this suggestion, we can calculate the first three moments of the total claims size distribution in the following way

$$\mu = nm = 415903,$$
$$\mu_2 = n\alpha_2 = 4{,}3 \times 10^9,$$
$$\mu_3 = n\alpha_3 = 1{,}1 \times 10^{14},$$

and for the coefficient of asymmetry we obtain

$$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} = 0{,}389.$$

If we limit the probability of insolvency to one percent ($\varepsilon = 1\% = 0{,}01$), then from the tables of Standard Normal distribution (Pollard 1977) we have $z_{0,01} = 2{,}33$. Substituting in (3) we obtain

$$100000 = (2{,}33 + \frac{0{,}389(2{,}33^2 - 1)}{6})\sqrt{4{,}3 \times 10^9} - 415903\,\lambda,$$

and we find $\lambda = 0{,}173$. Consequently, providing safety reserves over $17{,}3\%$ will limit the probability of insolvency in the end of the year to 1 percent.

Normal approximation in this case will lead to defining a lower percent of the safety loadings

$$100000 = 2.33 \times 65598 - 415903\lambda$$

Thus, the calculated result for the safety loadings is $\lambda = 12{,}7\%$. With such a result, using the Normal approximation may lead to a significant error of the estimation. If we express the safety loadings λ from equations (2) and (3), it can be seen that the Normal approximation will give a higher value of the safety loadings' estimation than the Normal power transformation if the following condition is fulfill

$$\gamma \frac{(z_\varepsilon - 1)}{6}\sqrt{n\alpha_2} < 0.$$

With the chosen probability of insolvency $\varepsilon = 1\%$, the inequality above will be fulfill only if $\gamma < 0$. When the coefficient of asymmetry is close to zero, then the difference of the results of the both methods wouldn't be so significantly.

MODELLING OF THE RETENTION

To use the fourth method for reducing the risk of insolvency is necessary to define a price, which may be reinsured.

Lets consider a reinsurer with reinsurer loss with retention M. If the distribution of the claims size das probability density function $s(x)$, then the second moment of the distribution of the individual claim size is

$$\alpha_2 = \int_0^M x^2 s(x)dx + M^2 \int_M^\infty s(x)dx \le M \int_0^M xs(x)dx + M^2 \int_M^\infty s(x)dx$$

$$\le M\left(\int_0^M xs(x)dx + M\int_M^\infty s(x)dx\right) \le Mm$$

where m is the mean of the individual claim size and thus we can write

$$\alpha_2 = k \times Mm,$$

where $k \le 1$ is a factor, which depends on the shape of the distribution of the claims.

If we ignore the safety reserves and write the total risk premium of the reinsurance as $P = mn$, then after substituting in (2) and (3) we obtain

$$U = z_\varepsilon \sqrt{kPM} - \lambda P,$$

where

$$M = \frac{(U + \lambda P)^2}{(z_\varepsilon)^2 kP}$$

(4)

or

$$U = (z_\varepsilon + \frac{\gamma_1 (z_\varepsilon^2 - 1)}{6})\sqrt{kPM} - \lambda P,$$

from where

$$M = \frac{(U + \lambda P)^2}{(z_\varepsilon + \frac{\gamma_1 (z_\varepsilon^2 - 1)}{6})^2 kP}$$

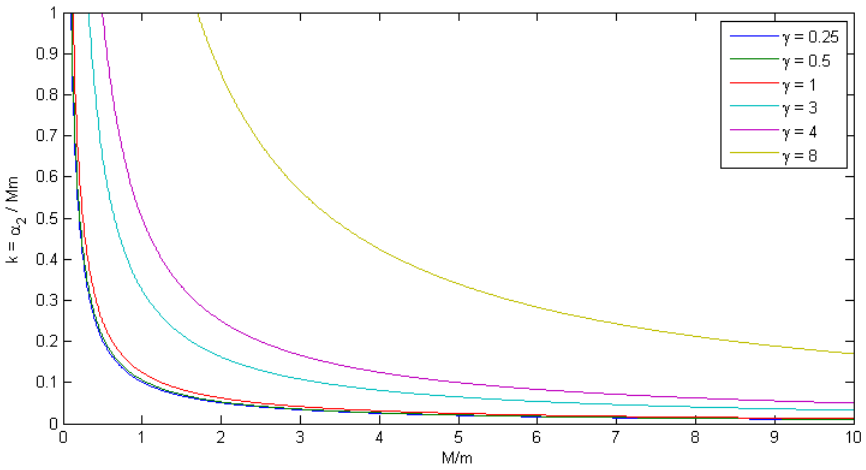

Figure 1 - Relation between α_2 and the product of $M \times m$

On Figure 1 is shown how the values of the factor k, vary depending on the second moment about the origin of the distribution of the individual claims α_2 and the product of $M \times m$. For illustration, some results with suggestions for Gamma distribution for the individual claims size and six different values for the coefficient of asymmetry γ are considered. We can notice that, for $\gamma \leq 1$, the value of k for the interval $[0,3; 1]$ decrease significially faster, compared with the cases when $\gamma > 1$. Such large difference is not observed with the lower values of k . For example, if we substitute $k = 0,25$ and choose $\varepsilon = 0,001$, from (4) we can express the relation between the retention M and free reserves U in the following way

$$\frac{M}{U} = \frac{c(1+\lambda\frac{P}{U})^2}{\frac{P}{U}}, \qquad (5)$$

where

$$c = \frac{1}{(z_\varepsilon)^2 k} \approx 0,42, \text{ при } k = 0,25, \ \varepsilon = 0,001.$$

From Figure 2 can be seen that, for a given value of the safety reserves λ, $\frac{M}{U}$ quickly decrease when $\frac{P}{U}$ increases up to a minimum and then starts to increase again. This minimum could be calculated from the first partial derivative of (4) by P, i.e. by solving the equation

$$\frac{\partial M}{\partial P} = cU(-\frac{U}{P^2} + \frac{\lambda^2}{U}) = 0$$

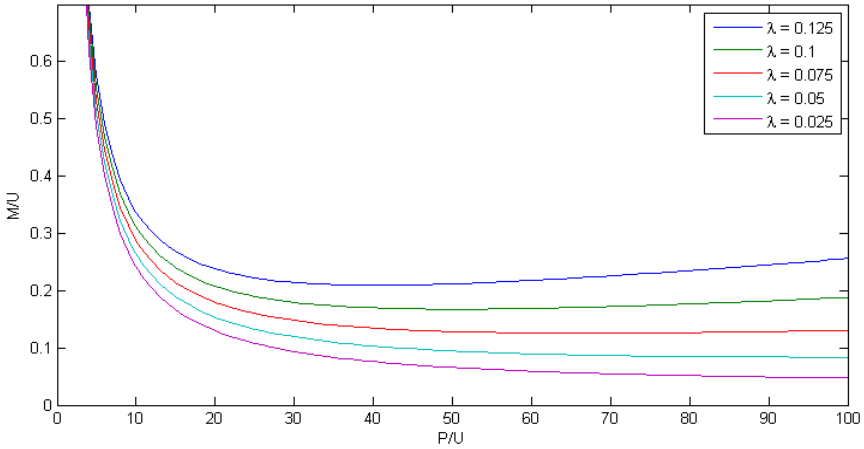

Figure 2 - Relation between retention and premium

The equation above is fulfilled when $P = \frac{U}{\lambda}$, from where we find

$$M_{min} = 4c\lambda U.$$

This is one method to determine the minimum price, which the insurer should reinsure.

MODERN RISK THEORY

In the Classical risk theory is assumed that, the claims are settled and paid immediately within the same year, in which they occur. In practice, in many insurance classes this does not happens. Furthermore, the stochastic nature of claims continues until they are reported. For this reason the level of solvency can never be defined precisely, it can be only estimated. These problems are in the scope of the Modern risk theory.

The insurer's attention is focused on the ratio of the risk premium and the actual gross premium payable (loss ratio)

$$l = \frac{P}{P'},$$

where P is the estimated risk premium and P' is the actual gross premium payable. The actuarial loss ratio includes additional insurer's expenses and safety loadings. Thus, if we express these expenses as a part of the net premium, we can write

$$P' = P + \lambda P + \nu P' = lP + \lambda P + \nu P'$$

Then the loss ratio could be expressed only with the insurer's expenses ν and the safety loadings λ

$$l = \frac{1-\nu}{1+\lambda}$$

The insurer's profit comes from two places: the insurer's profit, which is approximately equal to λP and from the investment income on shareholders' funds.

While in Classical risk theory, the size of the claims is the only random variable which is object of estimation, in Modern risk theory the solvency margin is the main object of interest (free reserves of shareholders funds).

In conventional insurance accounting terms, the limit of solvency margin can be expressed in the following way

$$T = Z_1 - Z_0 = P^{'} - C^{'} - E - I,$$

where:

T is the total profit before tax

Z_0 is the solvency margin at the start of the year

Z_1 is the solvency margin at the end of the year, before payment of tax and distribution of profit in respect of the year

$P^{'}$ is the earned premium ($= P + U_0 - U_1$)

P is the written premium

U_0 is the unearned premium provision at the start

U_1 is the unearned premium provision at the end

$C^{'}$ is the accounting incurred cost of claims ($= C + L_1 - L_0$)

C is the claim payments

L_0 is the outstanding claims provision at the start

L_1 is the outstanding claims provision at the end

E is the expenses of management, including commission

I is the investment earnings.

In actuarial terms, the insurance process is presented more detailed with some additional elements

$$T = P^{*} - C^{*} - E + K,$$

(6)

where:

P^* is the actuarial value of premiums earned ($= P + H + U_0 - U_1$)

H is the investment earnings on (or apportioned to) the unearned premium provisions

C^* is the actuarial incurred cost of claims ($= C - J + L_1 - L_0$)

J is the investment earnings on (or apportioned to) the outstanding claim provision

K is the investment earnings on (or apportioned to) the solvency margin

What should be insured is the level of solvency in the end of the year $\{Z_1 > Z_{min}\}$, which is equivalent to the probability ε, of the event $\{Z_1 < Z_{min}\}$ to be small, i.e.

$$Pr\{Z_1 < Z_{min}\} < \varepsilon.$$

The solvency margin at the start of the year could be considered as a constant and thus the last condition may be written as

$$Pr\{T_1 < T_{min}\} < \varepsilon \tag{7}$$

Standardizing the random variable T, (7) can be expressed in the following way

$$Pr\{\frac{T - E\{T\}}{\sqrt{var\{T\}}} < \frac{T_{min} - E\{T\}}{\sqrt{var\{T\}}}\} < \varepsilon,$$

whence

$$\frac{T_{min} - E\{T\}}{\sqrt{var\{T\}}} < F^{-1}(\varepsilon),$$

where $F^{-1}(x)$ is the inverse distribution function of standardized distribution of T. The point of risk estimation is the estimation of the variance of the total profit T, i.e. estimation of $var\{T\}$. Noting the fact that in practice the individual claims are never independent, from (6) we obtain

$$\text{var}\{T\} = \text{var}\{P^* - C^* - E + K\}$$
$$= \text{var}\{P^*\} + \text{var}\{C^*\} + \text{var}\{E\} + \text{var}\{K\} + \text{cov}\{P^*, C^*, E, K\}$$

Consequently, the variation of the total profit includes:

- variance arising in the experience of the current year's business, as estimated at the end of the year;
- variance arising in the development of past claims;

- variance arising in expenses;
- variance arising in the investment return;
- the correlation between these terms.

The variance, and respectively the risk can be considered in two aspects - independent and systematical. The independent risk is defined by the following elements:

- number of the potential claims, which will arise;
- reported claims;
- claims size;
- the run-off claim process.

The independent risk is proportional to the size of the portfolio of the shareholder. The systematical risk is proportional to the square of the size of the portfolio. The systematical risk is result of number of factors that affect portfolio's behavior, such as:

- long-term weather conditions, which can affect the frequency and severity of claims for a number of classes of insurance;
- variations in the investment market;
- inflation, which affects the amounts of claims in the liability classes;
- economic and social conditions;
- management and underwriting standards and others.

CONCLUSIONS

The choice of claim estimation method is of importance to when modelling insurance risk. Depending on the claims size distribution, the different methods may lead to differences in the estimation of the safety loadings. When the power of the coefficient of the distribution is high, these differences are tend to increase. When $\gamma < 0$ the Normal approximation leads to higher value for the estimation of the safety loadings. Vice versa - with positive coefficient of asymmetry $\gamma > 0$ the Normal power transformation will give the higher values for the safety loadings. If the form of the claim size distribution is symmetric, i.e. $\gamma \approx 0$, then the methods will result insignificantly different values. Defining the claim size distribution and the choice of method for the risk estimation is of a big importance for the results.

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