

STABILITY REGIONS FOR A DELAY COBWEB MODEL

Akio Matsumoto¹, Keiko Nakayama²

¹Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan

²Chukyo University, 101-2, Yagoto-Honmachi, Showa, Nagoya, Aichi, 466-8666, Japan

Abstract

This paper aims to study how an information lag on price affects economics dynamics. To this end, it constructs a delay cobweb model to conduct stability analysis with respect to the length of time delay. It is demonstrated that there is a threshold value of time delay and a steady state is asymptotically stable, loses stability and bifurcates to a limit cycle via Hopf bifurcation when a time delay is less than, equal to and larger than the threshold value.

Key words: information lag, cobweb dynamics, delay differential equation, stability switching curve, hopf bifurcation

1. INTRODUCTION

Dynamic model can be constructed in discrete and continuous time scales. It is well known that in examining dynamics, continuous models are more stable than discrete systems in a sense that the form has a larger stability region in the parameter space than the latter. Sometimes two models exhibit the opposite results. For example, Theocharis (1960) shows that the N -firm Cournot adjustment model in discrete time scale is unstable if $N \geq 3$ whereas the corresponding model in continuous time scale is always stable regardless of the number of firms. It is curious to know what makes these differences. As a first step, we consider a delay differential model that is a hybrid of these two models. Our main aim of this study is to provide stability analysis with respect to time delay and the behavioral parameters of the models. In particular we construct a simple 1-dimensional continuous model of price adjustment that is locally and globally stable. Introducing information lag to obtain the price information, we how the qualitative behavior of dynamics changes as the model parameters including the length of delay vary.

The investigation of stability for a delay differential equation can be reduced to the root location problem for the corresponding characteristic equation. Hayes (1950) determines stability conditions with respect to the model's parameter under which the real parts of the characteristic roots are all negative. Burger (1956) modifies Hayes' conditions to improve applicability. The complete stability region in the parameters space has already been characterized by Bellman and Cooke (1963) and Boese (1993), to name a few, while Kung (1993) summarizes the development of delay differential equations. The use of delay differential equations in the modeling of economic dynamics also has a long history. Haldane (1993) could be the first to examine economic dynamics in a delay differential equation. Kalecki (1935) and Goodwin (1951) investigate macroeconomic fluctuations by applying delay differential equations. It is still very active due to the fact that the delay is inevitable phenomenon in economics and also due to the recent mathematical developments for understanding the multiple delay system. In this paper, focusing on the simplest delay differential equation with constant coefficients, we analytically and geometrically determine the region of the behavioral parameters and the delay parameter for which the delay differential equation is stable or unstable.¹

¹ Matsumoto and Szidarovszky (2015) investigate the delay effects in the same Cobweb model with two different delays. Main focus is on the multiple delay effects on dynamics but no behavioral parameter effects are considered.

2. MODEL

Let $D(p)$ and $S(p^e)$ be the demand and supply functions of commodity. Here p is the commodity price and p^e is its expectation. For the sake of simplicity, these functions are assumed to be linear,

$$D(p) = d_1 - d_2 p, \quad d_1 > 0, \quad d_2 > 0 \quad (1)$$

and

$$S(p^e) = s_1 + s_2 p^e, \quad s_1 > 0, \quad s_2 > 0 \quad (2)$$

The equilibrium price p^* and quantity q^* satisfy the conditions of $p^* = p(t) = p^e(t)$ for all $t \geq 0$ and $q^* = D(p^*) = S(p^*)$ where

$$p^* = \frac{d_1 - s_1}{d_2 + s_2}$$

and

$$q = \frac{d_1 s_2 - d_2 s_1}{d_2 + s_2}$$

For positivity of the equilibrium price, we assume the following:

Assumption 1 $d_1 > s_1$.

We consider price dynamics in a continuous-time framework in which relative variations in market price at time t is adjusted to be proportional to the excess demand,

$$\frac{\dot{p}(t)}{p(t)} = K\{D[p(t)] - S[p^e(t)]\} \quad (3)$$

where $K > 0$ is the adjustment coefficient. Substituting equations (1) and (2) into (3) reduces the prices adjustment equation to the following form

$$\dot{p}(t) = Kp(t)[d_1 - s_1 - d_2 p(t) - s_2 p^e(t)]. \quad (4)$$

Equation (4) has two steady states, a trivial one $p = 0$ and a positive one, $p = p^*$. We are not concerned with the trivial solution, which will be eliminated from further considerations. Concerning the expectation formation, we take into account a spirit of the naïve expectation such that a past price at time $t - \tau$ is expected to be realized at time t .

Assumption 2 $p^e(t) = p(t - \tau)$, $\tau \geq 0$

3. DYNAMIC ANALYSIS

As a benchmark, we start with a non-delay case of $\tau = 0$. Substituting $p^e(t) = p(t)$ into the dynamic equation (4) yields an ordinary differential equation,

$$\dot{p}(t) = Kp(t)[d_1 - s_1 - (d_2 + s_2)p(t)]. \quad (5)$$

The equilibrium price is the only positive steady state. Arranging the terms can reduce equation (5) to the logistic equation,

$$\dot{p}(t) = \theta p(t) \left[1 - \frac{p(t)}{p^*} \right]$$

where $\theta = K(d_1 - s_1) > 0$. Solving this equation by separating the variables, we obtain

$$p(t) = \frac{p_0 p^*}{p_0 + (p^* - p_0)e^{-\theta t}}.$$

With the initial value $p_0 = p(0)$. This solution implies that the price approaches its equilibrium value as $t \rightarrow \infty$ for any initial value, so the equilibrium, is globally stable.

Theorem 1 *If supply is instantaneous (i.e., $\tau = 0$), then the equilibrium price of equation (5) is globally asymptotically stable with monotonic convergence.*

Since it is often observed in the real economy that the expectation formation is not instantaneous due to information delays, $\tau > 0$ is assumed henceforth. The dynamic equation then becomes a first-order nonlinear delay differential equation,

$$\dot{p}(t) = Kp(t)[d_1 - s_1 - d_2p(t) - s_2p(t - \tau)]. \quad (6)$$

If $d_2 = 0$ (i.e., perfectly inelastic demand), then equation (6) can be reduced to the Hutchinson equation or delay logistic equation,

$$\dot{p}(t) = \theta p(t) \left[1 - \frac{p(t - \tau)}{\bar{p}} \right]$$

with

$$\bar{p} = \frac{d_1 - s_1}{s_2} > 0$$

Note that $\bar{p} = p^*$ when $d_2 = 0$. Concerning stability of the positive steady state, the following results are well known.²

Theorem 2 *The positive steady state, $\bar{p}$ of equation (7) is asymptotically stable if $0 < \tau < \pi/2\theta$ and unstable if $\tau > \pi/2\theta$; a Hopf bifurcation occurs if $\tau = \pi/2\theta$ and a stable periodic exists if the steady state is unstable.*

We now proceed to the case where demand is imperfectly elastic (i.e., $0 < d_2 < \infty$) and see how the stability of the stationary point is affected when the values of delay and parameters are changed. If the right hand side of equation (6) is denoted by $G(p(t), p(t - \tau))$, then the linearized equation in a neighborhood of the stationary point $p_2^* = (p^*, p^*)$ is

$$\dot{p}_\delta(t) = \frac{\partial G}{\partial p(t)} \Big|_{p_2^*=(p^*, p^*)} p_\delta(t) + \frac{\partial G}{\partial p(t - \tau)} \Big|_{p_2^*=(p^*, p^*)} p_\delta(t - \tau)$$

or

$$\dot{p}_\delta(t) = -kd_2p_\delta(t) - ks_2p_\delta(t - \tau)$$

Where $k = Kp^* > 0$ and $p_\delta(t) = p(t) - p^*$. Introducing the new parameters $\alpha = kd_2 > 0$ and $\beta = ks_2 > 0$, we obtain the following form,

$$\dot{p}_\delta(t) + \alpha p_\delta(t) + \beta p_\delta(t - \tau) = 0. \quad (7)$$

Assuming an exponential solution

$$p_\delta(t) = e^{\lambda t} u$$

and substituting it into the linearized equation present the corresponding characteristic equation,

$$\lambda + \alpha + \beta e^{-\lambda \tau} = 0. \quad (8)$$

The equilibrium is asymptotically stable if all eigenvalues of equation (8) have negative real parts. Substituting $\lambda = \gamma + i\omega$ with $\omega \geq 0$ breaks down the characteristic equation into the real and imaginary parts,³

$$\gamma + \alpha + \beta e^{-\gamma \tau} \cos(\omega \tau) = 0, \quad (9a)$$

$$\omega - \beta e^{-\gamma \tau} \sin(\omega \tau) = 0. \quad (9b)$$

² See, for example, Kuang (1993) and Ruan (2006).

³ We will have the same results even if $\omega \leq 0$ is assumed.

If $\tau = 0$, then $\omega = 0$ via the equation (9b), $\gamma = -(\alpha + \beta) < 0$ via the equation (9a). This is another way of confirming the stability of the positive equilibrium in the case of no-delay. With the positive delay, we are interested in specifying parametric combination for which the stability could be lost. Let us denote the real part of eigenvalue λ by $Re\lambda$. It continuously depends on values of the delay as well as other parameters. We seek parametric conditions such as $Re\lambda = 0$. Since $\lambda = 0$ is not a solution, the characteristic equation must have a pair of purely imaginary solutions if $Re\lambda = 0$ holds. Supposing $\gamma = 0$ simplifies system (9a) and (9b),

$$\alpha + \beta \cos(\tau\omega) = 0, \quad (10a)$$

$$\omega - \beta \sin(\tau\omega) = 0. \quad (10b)$$

We can show that all pure complex roots of equation (8) are simple. Otherwise $\lambda = i\omega$ solves both equations

$$\lambda + \alpha + \beta e^{-\lambda\tau} = 0$$

and

$$1 + (-\tau)\beta e^{-\lambda\tau} = 0$$

implying that

$$1 + (\lambda + \alpha) = 0$$

which cannot occur if λ is a pure complex number.

We will obtain a threshold value of the delay for which stability might get lost. Before proceeding, we check the roles of the parameters α and β on dynamics. Returning to equation (7) and assuming $\beta = 0$, we have a solution

$$p_\delta(t) = p_\delta(0)e^{-\alpha t}$$

where $p_\delta(0)$ is an initial value. It monotonically converges to the 0 solution if $\alpha > 0$. Hence a positive α is a stabilizing factor. On the other hand, assuming $\alpha = 0$ and solving the two equations in (10) for τ yields a threshold value of the delay

$$\tau = \frac{\pi}{2\beta}.$$

A positive β is a destabilizing factor in a sense that a larger β makes the threshold value of τ smaller. Roughly speaking it is more likely that the model with a positive delay is more destabilized as β is larger.

Moving α and ω to the right hand side of the equations in system (10) and adding the squares of the resultant equations yield

$$\omega^2 = \beta^2 - \alpha^2.$$

This implies that ω is not an independent variable anymore. If $\alpha > \beta$, then there is no $\omega > 0$ and thus equation (8) has no purely imaginary solutions implying that no stability switch occurs. The stability of the equilibrium point is preserved for any value of τ . If $\alpha = \beta$, then we have $\omega = 0$. However, $\lambda = 0$ does not solve equation (8). Hence the stability is also preserved. A delay that does not affect stability is called *harmless*. So $\alpha \geq \beta$ intuitively implies that the stabilizing effect dominates over the destabilizing effect. This is a case in which any delay is harmless.

Theorem 3 *If $\alpha \geq \beta$, then the positive steady state of dynamic equation (6) is locally asymptotic stable for any positive value of τ .*

On the other hand, if $\alpha < \beta$, then we can obtain $\bar{\omega} > 0$ such as

$$\bar{\omega} = \sqrt{(\beta + \alpha)(\beta - \alpha)}. \quad (11)$$

There is a unique τ , $0 < \tau\bar{\omega} \leq 2\pi$ such that $\tau\bar{\omega}$ makes two equations in (10) hold. Thus solving

the first equation of system (10) yields the threshold value of τ ,

$$\bar{\tau}_m(\alpha, \beta) = \frac{1}{\bar{\omega}} \left[\cos^{-1} \left(-\frac{\alpha}{\beta} \right) + 2m\pi \right] (m = 0, 1, 2, \dots)$$

and solving the second equation gives

$$\bar{\tau}_n(\alpha, \beta) = \frac{1}{\bar{\omega}} \left[\pi - \sin^{-1} \left(\frac{\bar{\omega}}{\beta} \right) + 2n\pi \right] (n = 0, 1, 2, \dots) \quad (12)$$

Further, solving the first equation for β and substituting it into the second equation, then solving the resultant equation for τ yield the third expression for the critical value of the delay,

$$\bar{\tau}_k(\alpha, \beta) = \frac{1}{\bar{\omega}} \left[\pi - \tan^{-1} \left(\frac{\bar{\omega}}{\beta} \right) + 2k\pi \right] (k = 0, 1, 2, \dots)$$

Notice that $\bar{\tau}_m(\alpha, \beta)$, $\bar{\tau}_n(\alpha, \beta)$ and $\bar{\tau}_k(\alpha, \beta)$ give rise to the same value if $m = n = k$, although these have different forms. The $\tau = \bar{\tau}_0(\alpha, \beta)$ surface is illustrated in the (α, β, τ) space in Figure 1 with $\alpha \geq 0$, $\beta \geq 0$ and $\tau \geq 0$. The cube is divided into two triangular prisms by the plane standing on the $\alpha = \beta$ line. Since $\alpha \geq \beta$ holds in the right-hand triangular prism, the system is stable due to Theorem 3. The left-hand triangular prism is further divided into two parts by the convex-shaped $\tau = \bar{\tau}_0(\alpha, \beta)$ surface. It will be shown that the system is stable in the region below the surface (i.e., $\tau < \bar{\tau}_0(\alpha, \beta)$) and unstable above (i.e., $\tau > \bar{\tau}_0(\alpha, \beta)$).

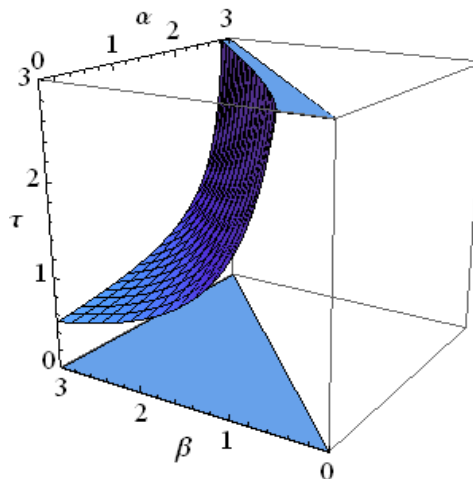


Figure 1. Stability changing surface in the (α, β, τ) space

We concentrate on finding the stability regions in the (α, τ) plane and the (β, τ) plane. To this end, the $\tau = \bar{\tau}_0(\alpha, \beta)$ surface is projected to the 2 dimensional plane to find the parameter effects on dynamics caused by a change in the parameter value. In Figure 2(A) the $\tau = \bar{\tau}_0(\alpha, \tilde{\beta})$ curve with $\tilde{\beta} = 5$ and the $\tau = \bar{\tau}_0(\tilde{\alpha}, \beta)$ curve with $\tilde{\alpha} = 1$ are, respectively, illustrated in the (α, τ) plane and the (β, τ) plane. In Figure 2(A) the $\tau = \bar{\tau}_m(\alpha, \tilde{\beta})$ curves for $m = 0, 1, 2, 3$ are depicted for $\alpha < \tilde{\beta}$. Each curve is positive-sloping as

$$\frac{\partial \bar{\tau}_m}{\partial \alpha} = \frac{1 + \alpha \bar{\tau}_m(\alpha, \tilde{\beta})}{\tilde{\beta}^2 - \alpha^q} > 0$$

and shift upward as m increases. It is confirmed that $\bar{\tau}_m(\alpha, \tilde{\beta})$ is asymptotic to the vertical line at $\alpha = \tilde{\beta}$ and

$$\bar{\tau}_m(0, \tilde{\beta}) = \frac{(1 + 4m)\pi}{2\tilde{\beta}}$$

and

$$\lim_{\alpha \rightarrow \tilde{\beta}} \bar{\tau}_m(\alpha, \tilde{\beta}) = \infty$$

In Figure 1, conditions $\beta \leq \tilde{\alpha}$ holds in the hatched area so that the system is stable due to Theorem 3. Next, we verify the shape of $\bar{\tau}_m(\tilde{\alpha}, \beta)$. Differentiating equation (12) with respect to β yields

$$\frac{\partial \bar{\tau}_m}{\partial \beta} = \frac{\tilde{\alpha} + \beta^2 \bar{\tau}_m(\tilde{\alpha}, \beta)}{\beta(\beta^2 - \tilde{\alpha}^2)} < 0 \quad (13)$$

Implying that the $\bar{\tau}_m(\tilde{\alpha}, \beta)$ curve is downward-sloping in the (β, τ) plane. Parameter m is increased from 0 to 3 and the corresponding four $\bar{\tau}_m(\tilde{\alpha}, \beta)$ curves are depicted. It is clear from equation (12) that increasing m shifts the $\bar{\tau}_m(\tilde{\alpha}, \beta)$ curve upward. It is also confirmed that each curve is asymptotic to the vertical line at $\beta = \tilde{\alpha}$ as $\beta \rightarrow \tilde{\alpha}$ and to the horizontal line at $2m\pi$ as $\beta \rightarrow \infty$, since we have

$$\lim_{\beta \rightarrow \tilde{\alpha}} \bar{\tau}_m(\tilde{\alpha}, \beta) = \infty \quad \text{and} \quad \lim_{\beta \rightarrow \infty} \bar{\tau}_m(\tilde{\alpha}, \beta) = 2m\pi.$$

We now examine the τ -effect generated by changes in τ . Since a solution of the characteristic equation is a continuous function of delay τ , differentiating equation (8) with respect to τ and rearranging terms yield

$$\frac{\partial \lambda}{\partial \tau} = \frac{\lambda \beta e^{-\lambda \tau}}{1 - \beta \tau e^{-\lambda \tau}}$$

Substituting $\lambda = i\omega$ and taking the real part give

$$\begin{aligned} \operatorname{Re} \left[\frac{\partial \lambda}{\partial \tau} \right]_{\lambda=i\omega} &= \operatorname{Re} \left[\frac{\lambda \beta e^{-\lambda \tau}}{1 - \beta \tau e^{-\lambda \tau}} \right]_{\lambda=i\omega} \\ &= \operatorname{Re} \left[-\frac{\lambda(\lambda + \alpha)}{1 + \alpha \tau + \tau \lambda} \right]_{\lambda=i\omega} = \operatorname{Re} \left[\frac{\omega^2 - i[\alpha \omega(1 + \alpha \tau) + \tau \omega^3]}{(1 + \alpha \tau)^2 + (\tau \omega)^2} \right] \end{aligned}$$

Hence

$$\frac{\partial(\operatorname{Re} \lambda)}{\partial \tau} \bigg|_{\lambda=i\omega} = \frac{\omega^2}{(1 + \alpha \tau)^2 + (\tau \omega)^2} > 0 \quad (14)$$

This inequality implies that all solutions crossing the imaginary axis at $i\omega$ cross from left to right as τ increases, that is, the real part becomes positive from negative.

Focusing on Figure 2(B), we see how the stability is lost when the value of τ increases. Given $\alpha > 0$, selecting β such as $\beta > \alpha$ and increasing the value of τ along the vertical line at this β , it sooner or later crosses the $\bar{\tau}_m$ curve with $m = 0$, the lowest downward-sloping curve in Figure 2(B). Let the τ -value of the intersection be $\bar{\tau}_0$. We have already confirmed that the equilibrium is stable for $\tau = 0$. Thus for $0 < \tau < \bar{\tau}_0$, all solutions of equation (8) have strictly negative real parts and thus the equilibrium is still stable under the positive delay. Since the selection of β is arbitrary as far as $\beta > \alpha$ holds, the equilibrium is stable in the region below the $\bar{\tau}_{m=0}$ curve. This region is colored in yellow in Figure 2(B). On the $\bar{\tau}_{m=0}$ curve, the real part of one solution becomes 0 and $\partial \operatorname{Re} \lambda / \partial \tau > 0$. The real part of this solution becomes positive and the equilibrium loses stability for $\tau > \bar{\tau}_0$. Further it can be shown by the Hopf bifurcation theorem that a periodic cycle emerges for $\tau > \bar{\tau}_0$. Stability is switched to instability at the first intersection. In other words, in the yellow region, the number of the positive roots is zero and the number increases to two in the region just above as

indicated by the figures in the parentheses in Figure 2(B). The $\bar{\tau}_{m=0}$ curve is the boundary between these two regions and we call it the *stability switching curve*. With further increasing τ , the vertical line crosses the $\bar{\tau}_m$ curve with $m = 1$, the second lowest curve. At the τ -value of this intersection, the real part of another solution becomes 0. No stability switch occurs at the second intersection (i.e., instability is preserved) but the number of the positive roots becomes four in the region just above this $\bar{\tau}_{m=1}$ curve. The same phenomenon can be observed at each intersection of the $\bar{\tau}_m$ curve with $m > 1$ and the number of the positive roots increases accordingly when we increase τ further. When τ increases and the value of β is fixed, the same phenomenon can be observed in Figure 2(A) in which the system is stable in the yellow region. We may go on from these observations to the following well-known results on the τ -effect: a larger delay has a destabilizing effect.

Theorem 4 If β and α are selected as $\beta > \alpha$, then the positive steady state of the dynamic equation (6) is

Asymptotically stable for $\tau < \bar{\tau}_0(\alpha, \beta)$

While it is

Unstable for $\tau > \bar{\tau}_0(\alpha, \beta)$

Where it loses stability at $\tau = \bar{\tau}_0(\alpha, \beta)$ and a limit cycle emerges for $\tau > \bar{\tau}_0(\alpha, \beta)$ with

$$\bar{\tau}_0(\alpha, \beta) = \frac{1}{\sqrt{(\beta + \alpha)(\beta - \alpha)}} \cos^{-1} \left(-\frac{\alpha}{\beta} \right) > 0$$

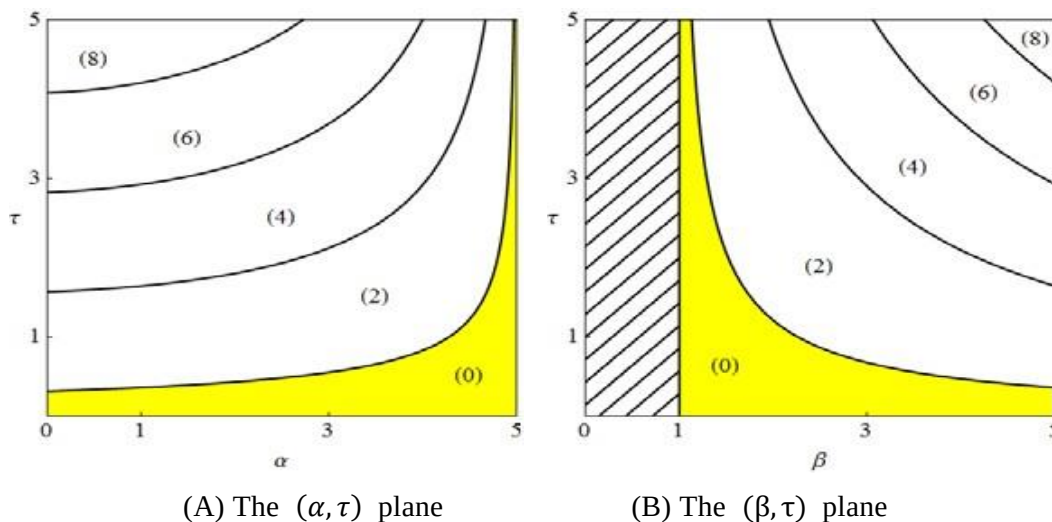


Figure 2. Stability switching curves in the 2D plane

Notice that $\beta \leq \alpha$ means $s_2 \leq d_2$, which then implies that the rate of increase of the supply is not greater than the rate of decrease of the demand. As is known, this is the stability condition of the cobweb model in discrete-time scale. It is already seen that the continuous-time cobweb model is always stable for any positive values of α and β . For $\beta > \alpha$, the discrete-time cobweb model is unstable while the delay cobweb model is stable for $\tau < \bar{\tau}_0$ and unstable otherwise. Hence the delay cobweb model is more stable than the discrete-time model and less stable than the continuous-time model.

4. CONCLUDING REMARKS

This study constructs a simple cobweb model with one time delay and conducts stability analysis with respect to the length of time delay τ and the behavioral parameters of the model, namely the slope of the demand curve, α and the slope of the supply curve β . It is shown that stability depends on α , β and τ in the following way:

- (1) When the demand curve is steeper than the supply curve (i.e., $\alpha \geq \beta$), time delay becomes harmless.
- (2) When the inequality is reversed (i.e., $\alpha < \beta$), there is a threshold value of the time delay and the model is asymptotically stable if the time delay is smaller than the threshold value and unstable if larger.
- (3) When the model is unstable, a periodic cycle emerges via Hopf bifurcation with respect to τ .

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