PARAMETER ESTIMATION IN WEIBULL DISTRIBUTION ON PROGRESSIVELY TYPE-II CENSORED SAMPLE WITH BETA-BINOMIAL REMOVALS

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Abstract

In this article, the estimation of parameters based on progressively type-II censored sample with random removals from the Weibull distribution is studied. The number of units removed at each failure time is assumed to follow a Beta-binomial distribution. Based on this type of censored sample, the maximum likelihood (ML) and approximate maximum likelihood (AML) estimators for the parameters of the Weibull distribution are derived. A Monte Carlo simulation study is also conducted to compare the performance of ML and AML estimators under progressively type-II censoring with the different random schemes.

Key words: *Weibull distribution, maximum likelihood estimator, approximate maximum likelihood estimator, progressively censoring type-II with random removals, beta-binomial removals*

1. INTRODUCTION

The Weibull distribution is one of the most commonly used distributions in the field of reliability and life-testing analysis where samples are usually censored. Among the different censoring schemes, the progressively Type-II censoring schemes has drawn attention by many authors during recent years. Because the traditional Type-I, Type-II and hybrid censoring schemes do not allow the experimenter to remove units before the termination of the experiment. Therefore, the progressive Type-II censoring schemes has been widely used in reliability and life-testing experiments, etc. The advantage of this censoring schemes can be explained (describe) as follows. The experimenter places n identical units on test at time zero and completely observe only m failures. When the first failure is observed, R_1 of the remaining $n-1$ surviving units are randomly selected and removed. Then after the second observed failure, R_2 of the remaining $n-1-R_1$ surviving units are randomly selected and removed, and so on. Finally, the experiment terminates until the m^{th} failure is observed and remaining $R_m = n-m-\sum_{i=1}^{m-1} R_i$ surviving units all removed. If $R_1 = R_2 = \cdots = R_{m-1} = 0$ then $R_m = n-m$ that corresponds Type-II censoring. If $R_1 = R_2 = \cdots = R_m = 0$, then $n = m$ that corresponds the complete sample. For a comprehensive recent review of progressive censoring, see (Balakrishnan & Aggarwala 2000) and (Balakrishnan 2007).

The $R_1, R_2, \ldots, R_m$ are all prefixed in this censoring scheme. However, in some practical situations, the R_i numbers randomly occur and so they cannot be prefixed. For instance, (Yuen & Tse 1996) indicated that the number of patients that withdraw from a clinical test at each stage is random and cannot be prefixed. (Kaushika et al. 2015) also point out that the probability of removal may vary from patient to patient and remains unknown to the experimenter. Thus, recently, the statistical inference for different lifetime distributions under progressive type II censoring with random removals has been investigated by various authors such as (Yuen & Tse 1996), (Wu et al. 2007), (Yan et al. 2011), and (Usta & Gezer 2015). However, there are few studies in the literature concerning the estimation of parameters from the Weibull distribution based on progressive type II censoring with random removals. For example, (Yuen & Tse 1996) considered the estimation problem when lifetimes are Weibull distributed and are collected under a Type-II progressive censoring with random removals follows a uniform discrete distribution. They derived the maximum likelihood (ML) estimator of the parameters and their asymptotic variances. Tse and Yuen (1998) provided the expected experiment times for Weibull-distributed lifetimes under Type- II progressive censoring, with the numbers of removals distributed binomial. Tse et. al. (2000) studied the analysis of Weibull distributed lifetime data observed under Type II progressive censoring with binomial removals. The ML estimators of the parameters and their asymptotic variances are derived. Tse and Xiang (2003) explored the problem of interval estimation for

parameters of Weibull distribution based on Type-II progressively censored with binomial random removals. In this paper, seven different confidence interval-estimation procedures were considered. Sarhan and Al-Ruzaizaa (2010) discussed statistical inference for the parameters of Weibull distribution model, using Type-II progressively censored data with binomial random scheme. They also used the ML method to derive both point and interval estimates of the parameters.

In this paper, we consider the estimation problem for two parameters of Weibull distribution under progressive Type II censored sample with random removals where the number of units removed at each failure time follows a Beta-binomial distribution. We derive the ML estimators of the unknown parameters. Since the ML estimators cannot be derived in explicit form, we obtain approximate maximum likelihood (AML) estimators, which have explicit expressions, for the parameters of Weibull. Moreover, a Monte Carlo simulation study is conducted to compare the performance of ML and AML estimators under progressively type-II censoring with the different random schemes.

The remainder of this paper is organized as follows. In Section 2, the Weibull distribution and the notation and definition used in this (throughout the) paper are presented. The ML and AML estimators are derived under Type-II progressive censoring with Beta-binomial removals in Sections 3 and 4, respectively. The results of the simulation study are presented in Section 5. The conclusions of the paper are provided in Section 6.

2. THE LIKELIHOOD FUNCTION OF THE MODEL

It is assumed that the lifetime X of a unit has a Weibull distribution with the shape (α) and scale (β) parameters. The probability density function (pdf) and the cumulative distribution function (cdf) of X are given as, respectively,

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}\ ; x>0, \alpha>0, \beta>0 \tag{2.1}$$

and

$$F(x;\alpha,\beta) = 1-e^{-\beta x^{\alpha}}\ ; x>0, \alpha>0, \beta>0 \tag{2.2}$$

Let $X_1 < X_2 < \cdots < X_m$ denote a progressively Type-II censored sample from a Weibull distribution, where $m < n$ is predetermined before the test. For progressive type II censoring with a pre-determined number of removals $R = (r_1, r_2, \ldots, r_m)$, the conditional likelihood function can be written as (Cohen 1963):

$$L_1(\Theta; x|R=r) = C\prod_{i=1}^{m} f(x_i)[1-F(x_i]^{r_i} \tag{2.3}$$

where, $C = n(n-m-r_1)\ldots(n-m-\sum_{i=1}^{m} r_i+1)$. Substituting Eqs. (2.1) and (2.2) into Eq. (2.3), the conditional likelihood function is derived as:

$$L_1(\alpha,\beta; x|R=r) = C\prod_{i=1}^{m} \alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}\,(e^{-\beta x^{\alpha}})^{r_i} \tag{2.4}$$

Suppose that the number of units removed at each failure time R_i, follows a binomial distribution with parameters $n-m-\sum_{j=1}^{i-1} r_j$ and p. Thus, for given p, the probability of R_i units leaving after the i^{th} failure is given by:

$$P(R_i=r_i|p) = \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} p^{r_i}(1-p)^{n-m-\sum_{j=1}^{i} r_j}\ ; i=1,2,\ldots,m-1 \tag{2.5}$$

Also, similar to (Singh et al. 2013) and (Kaushik et al. 2015), we assumed that the probability of removals (p) is not fixed and p is a random variable following a beta distribution with parameters a and b:

$$g(p|a,b) = \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1}\ ;\ a,b>0, 0<p<1 \tag{2.6}$$

Then, the distribution of R_i can be obtained as follows

$$P(R_i = r_i, a, b) = \int_0^1 P(R_i = r_i|p) g(p|a,b)\, dp \tag{2.7}$$

Substituting Eqs. (2.5) and (2.6) in Eq. (2.7) and after simplification, we get

$$P(R_i = r_i, a, b) = \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} \frac{B(a+r_i, b+n-m-\sum_{j=1}^{i} r_j)}{B(a,b)} \tag{2.8}$$

where, $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $a,b>0$ and $r_i = 0,1,\dots,n-m-\sum_{j=1}^{i-1} r_j,\ i = 1,2,\dots,(m-1)$. The probability mass function given in Eq. (2.8) is known as Beta-binomial distribution and it is denoted by $Beta-Binom(n^*,a,b)$ (Singh et al. 2013). The sampling procedure for generating a progressively Type-II censored sample with Beta-binomial removals from a test, is illustrated in Table 1.

The joint probability of $R_1 = r_1, R_2 = r_2, \dots, R_m = r_m$ is given by

$$P(R = r, a, b) = P[R_1 = r_1]\ x\ P[R_2 = r_2|R_1 = r_1]\ x\ \dots$$
$$x\ P\left[R_{m-1} = r_{m-1}|R_{m-2} = r_{m-2,\dots,} R_1 = r_1\right] \tag{2.9}$$

then

$$P(R = r, a, b) = A \frac{1}{(B(a,b))^{m-1}} \prod_{i=1}^{m-1} B(a+r_i, b+n-m-\sum_{j=1}^{i} r_j) \tag{2.10}$$

where $A = \frac{(n-m)!}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{j=1}^{m-1} r_j)!}$.

Moreover, we presume that R_i is independent of X_i for all i^{th}. Accordingly, the likelihood function can be expressed as

$$L(\alpha,\beta,a,b;x,r) = L_1(\alpha,\beta;x|R=r)P(R=r,a,b) \tag{2.11}$$

Table 1: Sampling procedure for a life test under progressive Type-II censoring scheme with Beta-binomial removals

Stage	*Failed unit*	*Removed units*	*Survived units*
1	x_1	$r_1 \sim BetaBinom(n-m, a, b)$	$n-1-r_1$
2	x_2	$r_2 \sim BetaBinom(n-m-r_1, a, b)$	$n-2-r_1-r_2$
.	.	.	.
.	.	.	.
.	.	.	.
i	x_i	$r_i \sim BetaBinom(n-m-\sum_{j=1}^{i-1} r_j, a, b)$	$n-i-\sum_{j=1}^{i-1} r_j$
.	.	.	.
.	.	.	.
.	.	.	.
m-1	x_{m-1}	$r_{m-1} \sim BetaBinom(n-m-\sum_{j=1}^{m-2} r_j, a, b)$	$n-m+1-\sum_{j=1}^{i-1} r_j$
m	x_m	All-Out	0

3. MAXIMUM LIKELIHOOD ESTIMATIONS

In this section, we derive the maximum likelihood estimators (MLEs) of the parameters α, β and a, b under progressively Type II censoring sample with random removals where the number of units removed at each failure time follows a Beta-binomial distribution.

It is clear that $L_1(\alpha, \beta)$ does not include a and b .Thus, the MLEs of α and β can be derived by maximizing Eq. (2.4) directly. The log likelihood of $L_1(\alpha, \beta)$ is written as:

$$l_1(\alpha, \beta; x|R=r) = lnC + mln\alpha + mln\beta - \beta \sum_{i=1}^{m} x_i^{\alpha}(1+r_i) + (\alpha - 1)\sum_{i=1}^{m} lnx_i \qquad (3.1)$$

The MLEs of α and β, can be obtained by solving the following normal equations:

$$\frac{\partial l_1(\alpha, \beta; x|R=r)}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m} x_i^{\alpha}(1+r_i) = 0 \qquad (3.2)$$

$$\frac{\partial l_1(\alpha, \beta; x|R=r)}{\partial \alpha} = \frac{m}{\alpha} - \beta \sum_{i=1}^{m} x_i^{\alpha}(1+r_i) lnx_i + \sum_{i=1}^{m} lnx_i = 0 \qquad (3.3)$$

It should be noted that there is no explicit solution for Eqs. (3.2) - (3.3), and thus, iterative methods should be used [34, 36]. In this paper, we use the well-known Newton-Rapson to obtain the MLEs $\hat{\alpha}$ and $\hat{\beta}$ of α and β.

Similarly, because of $P(R=r, a, b)$ does not involve α and β, the MLEs of a and b can be derived by maximizing Eq. (2.10) directly. The log likelihood of $P(R=r, a, b)$ is written as

$$\log P = lnA - (m-1)[ln\,\Gamma(a) + \ln\Gamma(b) - \ln\Gamma(a+b)] + \sum_{i=1}^{m-1} ln\Gamma(a+r_i) + \sum_{i=1}^{m-1} ln\Gamma(n-m-\sum_{j=1}^{i} r_j + b) - \sum_{i=1}^{m-1} ln\Gamma\left(n-m-\sum_{j=1}^{i-1} r_j + a + b\right) \qquad (3.4)$$

The MLEs $\hat{a}$ and $\hat{b}$ of a and b can be obtained by solving the normal equations $\frac{d\log P}{da} = 0$ and $\frac{d\log P}{db} = 0$. However, we will not concerned with that in this paper.

4. APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATION

Since the ML estimators of α and β do not obtain as explicit formulas for the Weibull distribution based on progressive Type II censored sample with random removals, we derive the approximate maximum likelihood (AML) estimators developed by Balakrishnan (1989) and (Balakrishnan & Vardan 1991) to estimate the parameters α and β.

As mentioned before, since the probability $P(R = r, a, b)$ does not depend on the parameters α and β, so we ignore it and if we consider the transformation $V_i = \ln X_i$, $i = 1,2, \dots, m$ and $\alpha = \frac{1}{\sigma}$ and $\mu = -\frac{1}{\alpha} ln\beta$. Then, the likelihood function can be written as:

$$L_1(\sigma, \mu | R = r) = C \prod_{i=1}^{m} \frac{1}{\sigma} e^{\frac{v_i - \mu}{\sigma} - e^{\frac{v_i - \mu}{\sigma}}} (e^{\frac{v_i - \mu}{\sigma}})^{r_i} \quad (4.1)$$

where $C = n(n - m - r_1) \dots (n - m - \sum_{i=1}^{m} r_i + 1)$.

Let $z_i = \frac{v_i - \mu}{\sigma}$, $g(z_i) = e^{z_i - e^{z_i}}$ and $\bar{G}(z_i) = e^{-e^{z_i}}$, then the Eq. (4.1) can be rewritten as:

$$L_1(\sigma, \mu | R = r) = c \prod_{i=1}^{m} \frac{1}{\sigma} g(z_i)(\bar{G}(z_i))^{r_i} \quad (4.2)$$

with the log-likelihood function

$$l_1(\sigma, \mu; x | R = r) = lnc - mln\sigma + \sum_{i=1}^{m} g(z_i) + \sum_{i=1}^{m} r_i \bar{G}(z_i) \quad (4.3)$$

Taking partial differentiation of $l_1(\sigma, \mu; x | R = r)$ with respect to μ and σ, the normal equations are given following as:

$$\frac{dl_1(\sigma, \mu; x | R = r)}{d\mu} = -\frac{1}{\sigma} \sum_{i=1}^{m} \frac{g'(z_i)}{g(z_i)} + \sum_{i=1}^{m} \frac{r_i}{\sigma} \frac{g(z_i)}{\bar{G}(z_i)} = 0 \quad (4.4)$$

$$\frac{dl_1(\sigma, \mu; x | R = r)}{d\sigma} = -\frac{m}{\sigma} - \sum_{i=1}^{m} \frac{g'(z_i)}{g(z_i)} \frac{z_i}{\sigma} + \sum_{i=1}^{m} r_i \frac{g(z_i)}{\bar{G}(z_i)} \frac{z_i}{\sigma} = 0 \quad (4.5)$$

Clearly, the normal equations (4.4) and (4.5) do not have explicit solutions. Thus, we consider a first order Taylor expansion to the function $\frac{g'(z_i)}{g(z_i)}$ and $\frac{g(z_i)}{\bar{G}(z_i)}$ around the $G^{-1}(p_i) = \ln(-lnq_i) = \mu_i$ where $p_i = \frac{i}{m+1}$, $q_i = 1 - p_i$, $i = 1,2, \dots, m$ see (Balakrishnan & Aggarwala 2000) and (Hashemi & Amiri 2011).

Then, we can consider the following approximations:

$$\frac{g'(z_i)}{g(z_i)} \approx \alpha_i - \beta_i z_i \quad (4.6)$$

$$\frac{g(z_i)}{\bar{G}(z_i)} \approx 1 - \alpha_i + \beta_i z_i \quad (4.7)$$

where $\alpha_i = 1 + lnq_i(1 - \ln(-lnq_i))$ and $\beta_i = -lnq_i$. By using Eqs. (4.6) and (4.7) into Eqs. (4.4) and (4.5), we get

$$-\sum_{i=1}^{m} (\alpha_i - \beta_i z_i) + \sum_{i=1}^{m} r_i (1 - \alpha_i + \beta_i z_i) = 0 \quad (4.8)$$

$$-m-\sum_{i=1}^{m}(\alpha_i-\beta_i z_i)z_i+\sum_{i=1}^{m}r_i\,(1-\alpha_i+\beta_i z_i)z_i=0 \tag{4.9}$$

From Eqs. (4.8) and (4.9), we can obtain

$$\mu = B - D\,\sigma \tag{4.10}$$

$$m\sigma^{\,2}+E\sigma\,-F=0 \tag{4.11}$$

where,

$$B=\frac{\sum_{i=1}^{m}(r_i+1)\beta_i z_i}{\sum_{i=1}^{m}(r_i+1)\beta_i},\qquad D=\frac{\sum_{i=1}^{m}\alpha_i-\sum_{i=1}^{m}r_i\,(1-\alpha_i)}{\sum_{i=1}^{m}(1+r_i)\beta_i},$$

$$E=\sum_{i=1}^{m}\alpha_i(x_i-B)-\sum_{i=1}^{m}r_i(1-\alpha_i)\,(x_i-B)-2D\sum_{i=1}^{m}(1+r_i)\,\beta_i(x_i-B),$$

$$F=\sum_{i=1}^{m}(1+r_i)\,\beta_i(x_i-B)^2\,.$$

Thus, AML estimators of σ and μ are, respectively, $\hat{\sigma}=\frac{-E+\sqrt{E^2+4Fm}}{2m}$ and $\hat{\mu}=B-D\hat{\sigma}$. Then, we obtain $\hat{\alpha}=1/\hat{\sigma}$ and $\hat{\beta}=e^{-\hat{\alpha}\hat{\mu}}$ as the AMLE of α and β.

5. SIMULATION

In this section, a Monte Carlo simulation study is conducted to compare the performance of the ML and AML estimates derived in the previous sections for progressively Type II censoring data with the different Beta-binomial random schemes.

5.1 Algorithm for generating progressively type-II censored samples from Weibull distribution

By applying the algorithms given in (Balakrishnan & Aggarwalla 2000) and (Singh et al. 2013), the following steps are used generate progressively Type II censored data with the Beta-binomial removals from Weibull distribution.

The steps are:

1. Fix the values of $n, m,$.
2. Fix the values of parameters α, β and a, b.
3. Generate a random number r_1 from $Beta-Binom(n-m,a,b)$.
4. Generate a random number r_i from $Beta-Binom\left(n-m-\sum_{j=1}^{i-1}r_j\,,a,b\right);\ i=2,\dots,m-1$
5. Set $r_m=\begin{cases}n-m-\sum_{j=1}^{m-1}r_j & if\ \ n-m-\sum_{j=1}^{m-1}r_j\ >0\\ 0 & otherwise.\end{cases}$
6. Generate m independent $w_i;\ \ i=1,2,\dots,m$ from $Uniform(0,1)$.
7. Set $V_i=w_i^{\,1/\left(i+\sum_{j=m-i+1}^{m}r_j\right)}\ \ \forall\ \ i=1,2,\dots,m.$
8. Set $U_i=1-V_mV_{m-1}\dots V_{m-i+1}\ \ \forall\ \ i=1,2,\dots,m.$ Then, $U_1,U_2,\dots,U_m$ are ranked progressive censored sample of size m from $Uniform(0,1)$ with Beta-binomial removals.
9. Finally, for fixed values of parameters α and β, we set $X_i=1-F^{-1}(U_i)=\left(\frac{-\ln(1-U_i)}{\beta}\right)^{1/\alpha}$. Then, $(X_1,X_2,\dots,X_m)$ is the progressive type-II censored sample from the Weibull distribution with the Beta-binomial removals.

5.2 Simulation design

- Given the values of parameters $\beta = 1;\ \alpha = 1.5, 2$ and $\beta = 2;\ \alpha = 2$, sample size n, the number of failure m and the values of parameters a and b, generate a progressively type-II censored sample using the algorithms in Section 5.1. The sample size n is taken as 20, 30 and 50, the value of percentiles m failures are taken as $\left(\frac{m}{n}\right) x100 = 40\%$ and 80% and the values of parameters (a, b) are considered as (1,1) and (2,2).
- Compute the ML and AML estimates of parameters α and β.
- Repeat first and second steps 5000 times.
- Compare the performance of the ML and AML estimators by computing bias and MSE.

5.3 Simulation results

The obtained results from the simulation study are summarized in Tables 2-3. First, table 2 provides the bias and MSE of the ML and AML estimators for $\beta = 1, \alpha = 1.5$ and $a = 1, b = 1\ ,\ a = 2\ , b = 2$.

Table 2: Simulation results for $\beta = 1, \alpha = 1.5$

		MLE		AMLE			MLE		AMLE	
		$a = 1$		$b = 1$			$a = 2$		$b = 2$	
		$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$		$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$
n=20 m=8	**MSE**	0,2177	0,2409	0,2624	0,1666		0,2143	0,2368	0,2590	0,1660
	Bias	0,1405	0,1913	0,2674	-0,0105		0,1493	0,1917	0,2694	-0,0118
n=20 m=16	**MSE**	0,0996	0,1303	0,1105	0,1193		0,0912	0,1346	0,1022	0,1249
	Bias	0,0585	0,1357	0,1020	0,1028		0,0605	0,1290	0,1038	0,0981
n=30 m=12	**MSE**	0,1397	0,1343	0,1677	0,1050		0,1439	0,1367	0,1719	0,1045
	Bias	0,1090	0,1256	0,2099	-0,0379		0,1179	0,1309	0,2153	-0,0299
n=30 m=24	**MSE**	0,0570	0,0776	0,0618	0,0728		0,0598	0,0742	0,0650	0,0698
	Bias	0,0356	0,0853	0,0664	0,0620		0,0392	0,0892	0,0698	0,0662
n=50 m=20	**MSE**	0,0793	0,0714	0,0929	0,0640		0,0725	0,0662	0,0856	0,0618
	Bias	0,0664	0,0795	0,1417	-0,0432		0,0629	0,0663	0,1361	-0,0538
n=50 m=40	**MSE**	0,0311	0,0405	0,0330	0,0390		0,0306	0,0390	0,0324	0,0375
	Bias	0,0207	0,0524	0,0402	0,0377		0,0190	0,0488	0,0384	0,0341

The results given in Table 2 point out that for β and α, the MSEs of MLEs and AMLEs decrease when the sample size n and the failure information m increase under all censoring random schemes. The MLEs overestimate β and α. However, the AMLE underestimate α except for n=50,m=40 while β is overestimated by the AMLE. For β, the MLE provides minimum biases when all the sample sizes and censoring random schemes. The AMLE exhibits the best performance for α in terms of biases. With respect to the MSEs, for all the sample sizes and censoring random schemes, while the MLE gives the best estimates for β, the AMLE is the best estimator for α.

Table 3 reports the bias and MSE of the ML and AML estimators for $\beta = 1, \alpha = 2$ and $a = 1, b = 1\ ,\ a = 2\ , b = 2$.

Table 3: Simulation results for $\beta = 1, \alpha = 2$

		MLE		AMLE			MLE		AMLE	
		$a = 1$		$b = 1$			$a = 2$		$b = 2$	
		$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$		$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$
n=20 m=8	**MSE**	0,2105	0,4333	0,2613	0,3005		0,2027	0,4158	0,2410	0,2911
	Bias	0,1450	0,2551	0,2728	-0,0145		0,1423	0,2491	0,2608	-0,0229
n=20 m=16	**MSE**	0,0947	0,2376	0,1059	0,2190		0,0976	0,2307	0,1092	0,2129
	Bias	0,0641	0,1816	0,1074	0,1380		0,0657	0,1693	0,1092	0,1249
n=30 m=12	**MSE**	0,1437	0,2303	0,1749	0,1852		0,1465	0,2306	0,1732	0,1808
	Bias	0,1122	0,1558	0,2146	-0,0575		0,1217	0,1628	0,2178	-0,0574
n=30 m=24	**MSE**	0,0596	0,1418	0,0649	0,1330		0,0589	0,1305	0,0643	0,1223
	Bias	0,0408	0,1211	0,0714	0,0899		0,0423	0,1061	0,0726	0,0755
n=50 m=20	**MSE**	0,0723	0,1210	0,0863	0,1106		0,0725	0,1212	0,0851	0,1078
	Bias	0,0650	0,0976	0,1404	-0,0669		0,0623	0,1059	0,1362	-0,0598
n=50 m=40	**MSE**	0,0322	0,0722	0,0343	0,0695		0,0307	0,0713	0,0324	0,0685
	Bias	0,0262	0,0654	0,0457	0,0458		0,0178	0,0666	0,0374	0,0463

It can be seen from Table 3 that similar to results of $\beta = 1, \alpha = 1{,}5$, when $\beta = 1, \alpha = 2$, the MLE performs better than the AMLE for β according to the biases and MSEs when all the sample sizes and censoring random schemes. For α, the biases and MSEs of the AMLE are smaller than the MLE for all considered censoring random schemes and sample sizes.

Table 4 provides the bias and MSE of the ML and AML estimators for $\beta = 2, \alpha = 2$ and $a = 1$, $b = 1$, $a = 2$, $b = 2$.

Table 4: Simulation results for $\beta = 2, \alpha = 2$

		MLE		AMLE			MLE		AMLE	
		$a = 1$		$b = 1$			$a = 2$		$b = 2$	
		$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$		$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$
n=20 m=8	**MSE**	0,2023	0,2184	0,1676	0,2150		0,1870	0,2219	0,1582	0,2200
	Bias	-0,1898	0,0565	-0,0846	-0,1831		-0,1809	0,0647	-0,0850	-0,1782
n=20 m=16	**MSE**	0,1182	0,1640	0,1135	0,1560		0,1195	0,1624	0,1143	0,1541
	Bias	-0,1015	0,0853	-0,0460	0,0447		-0,1066	0,0843	-0,0507	0,0450
n=30 m=12	**MSE**	0,1421	0,1420	0,1205	0,1560		0,1428	0,1453	0,1243	0,1566
	Bias	-0,1334	0,0399	-0,0600	-0,1593		-0,1218	0,0484	-0,0559	-0,1552
n=30 m=24	**MSE**	0,0932	0,1107	0,0919	0,1056		0,0911	0,1035	0,0906	0,0996
	Bias	-0,0540	0,0707	-0,0143	0,0407		-0,0529	0,0615	-0,0134	0,0318
n=50 m=20	**MSE**	0,0964	0,0852	0,0865	0,1008		0,1003	0,0839	0,0894	0,1006
	Bias	-0,0661	0,0339	-0,0202	-0,1251		-0,0718	0,0260	-0,0284	-0,1313
n=50 m=40	**MSE**	0,0674	0,0639	0,0673	0,0626		0,0687	0,0605	0,0685	0,0593
	Bias	-0,0124	0,0406	0,0123	0,0214		-0,0210	0,0391	0,0037	0,0192

It can be concluded from Table 4 that the MLEs underestimate β, but it overestimate α for when all the sample sizes and censoring random schemes. The AMLE underestimate β except for $n = 50, m = 40$ while α is underestimated by the AMLE when the rate of m failures in the n sample is %40. According to biases, for β, the AMLE shows the best performance for all the sample sizes and censoring random schemes. However, the MLE provides minimum biases for α when the rate of m failures in the n sample is 0.4. With respect to the MSEs, the AMLE gives the best result for β when all the sample sizes and censoring random schemes, but the MLE works well for α when $n = 30, m = 12$ and $n = 50, m = 20$.

6. CONCLUSION

In this study, we have considered the problem of parameter estimation for Weibull distribution in presence of progressively Type-II censored sample with Beta-Binomial removals. For this purpose, the maximum likelihood (ML) and approximate maximum likelihood (AML) estimators for the parameters of the Weibull distribution based on this type of censored sample are derived. Furthermore, the performance of ML and AML estimators is compared in terms of bias and MSE for different sample sizes and random schemes via a Monte Carlo simulation.

As a consequence, the overall simulation results reveal the following:

- When the values of parameters are $\beta = 1;\ \alpha = 1.5, 2$, for all the sample sizes and censoring random schemes, the MLE performs better than the AMLE in terms of bias and MSE for β and the AMLE provides minimum biases and MSEs for α.
- If the values of parameters are $\beta = 2;\ \alpha = 2$, considering bias and MSE, the AMLE is exhibits best performance for β when all the sample sizes and censoring random schemes. the AMLE gives mostly good estimates for α. However, the MLE provides minimum biases and MSE for only α when the rate of m failures in the n sample is 0.4.

- Based on this, the AML is generally more efficient than ML to estimate the parameters β and α of Weibull distribution under progressively Type-II censored sample with Beta-Binomial removals.

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