

**IRREDUCIBLE DECOMPOSITION OF ELASTIC CONSTANT TENSOR
FOR ANISOTROPIC ENGINEERING MATERIALS**

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Abstract

A new irreducible decomposition method for elastic constant tensor is presented. Harmonic and other existing decomposition procedures in literature are elaborated and compared which have many engineering and scientific applications for anisotropic materials. For comparison purposes, numerical examples of each decomposition process are presented for the materials possessing different anisotropic elastic symmetry classes. For isotropic materials, an expression for the elastic constant tensor different than the traditional known form is given. Some misprints found in the literature are corrected. As a result of comparison process, it is also shown that total scalar (isotropic) part is decomposed into two physically meaningful orthogonal parts by irreducible decomposition, while in harmonic decomposition method, decomposition of total scalar part is not orthogonal. In this work, it is proved that these parts can be made orthogonal to each other. This case is a significant innovation in decomposition procedures in literature.

Key words: *elastic constant tensor, irreducible decomposition method, harmonic decomposition method, anisotropic elastic symmetries, orthogonality.*

1. INTRODUCTION

A material is isotropic if its mechanical and elastic properties are the same in all directions. When this is not true, the material is anisotropic. Anisotropic materials become the material of choice in a variety of engineering applications in the last century. Many materials are anisotropic and inhomogeneous due to the varying composition of their constituents. For instance, polycrystalline materials generally show an elastic anisotropy due to texture and the anisotropy of single crystallites. The polycrystalline and composite materials which show high anisotropy are used in many applications in industry. Everyday passed, the number of anisotropic materials is increasing by the addition of man-made anisotropic single crystals and technologically developed anisotropic materials. In order to understand the physical properties of the anisotropic materials, use of tensors by decomposing them is important. Tensors are the most significant mathematical entities to describe direction dependent physical properties of solids and the tensor components characterizing physical properties which must be specified without reference to any coordinate system.

The anisotropic form of Hooke's law in linear elasticity is often written in indicial notation as

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad (1)$$

where σ_{ij} are components of stress tensor, ε_{kl} are components of infinitesimal strain tensor and C_{ijkl} are the components of elasticity tensor and satisfy three important symmetry restrictions. These are

$$C_{ijkl} = C_{jikl} \quad C_{ijkl} = C_{ijlk} \quad C_{ijkl} = C_{klij} \quad (2)$$

which follow from the symmetry of the stress tensor, the symmetry of the strain tensor and the elastic strain energy. These restrictions reduce the number of independent elastic constants C_{ijkl} from 81 to 21 (see for example Nye, 1957).

The indices are abbreviated according to the replacement rule given in the following table:

Table 1. Abbreviation of indices for four index and double index notations (Nye (1957)).

Four index notation	11	22	33	23,32	13,31	12,21
Double index notation	1	2	3	4	5	6

Studies of irreducible decomposition began with Schouten (1951) who named the irreducible parts as scalar, vector, deviator, septor and nonor. Sirotnin (1975) decomposed tensors into irreducible parts in two stages; first with respect to the linear group and then with respect to the orthogonal group $O(3)$. Jerphagnon et al. (1978) derived certain results for the irreducible tensors in their natural form. Andrews and Ghoul (1982) followed the technique of Jerphagnon *et al.* and gave the reduction of a fourth rank cartesian tensor into irreducible parts under the three-dimensional rotation group.

For harmonic decomposition method, the works are summarized as; Backus (1970) proposed a representation of elastic constant tensor in terms of harmonic tensors. These are based on an isomorphism between the space of homogeneous harmonic polynomials of degree q and the space of totally symmetric tensors of order q . Baerheim (1993) followed Backus (1970) and developed the method. There are also other works for decomposition of elastic constant tensor, to name some, Onat (1984), Cowin (1989), Onat et al. (1994), Forte and Vianello (1996, 2006), Ting and He (2006), Zheng (2007), Annin and Ostrosablin (2008).

The aims of this work are to develop a new irreducible decomposition method which gives physically meaningful orthogonal parts, instead of other methods like harmonic decomposition in literature and also prove that the decomposed parts obtained from these methods can be made orthogonal to each other.

In present paper, following the works of Jerphagnon *et al.* (1978) and Andrews and Ghoul (1982), a new procedure for irreducible decomposition of elastic constant tensor is described. Harmonic and other decomposition methods are introduced. As applications for them, numerical examples are given. Finally, the methods are compared and discussed. Conclusions pertinent to this work are also stated.

2. DECOMPOSITION OF ELASTIC CONSTANT TENSOR

2.1. By irreducible decomposition method

Any rank- n cartesian tensor can be written as the direct sum of irreducible tensors in the cartesian representation. The term irreducible indicates sets that cannot be resolved into subsets with separate

linear transformations. The reduction of a (rank- n) cartesian tensor $\mathbf{T}_{(n)}$ generally results in a sum of irreducible tensors, with some weights (j) represented more than once. (where $0 \leq j \leq n$), it can be accomplished by the formula: $\mathbf{T}_{(n)} = \sum_{j=0}^n \sum_{q=1}^{N_n^{(j)}} \mathbf{T}_{(n)}^{(j;q)}$, where q is called the seniority index of the irreducible tensor $\mathbf{T}_{(n)}^{(j;q)}$ (irreducible cartesian tensor which is symmetric and traceless) and $N_n^{(j)}$ is the multiplicity of weight j in this reduction, it denotes the number of independent weight- j irreducible tensor parts. (See for instance, Jerphagnon et al. (1978) and Andrews, Ghoul (1982))

$$N_n^{(j)} = \sum_k (-1)^k \binom{n}{k} \binom{2n-3k-j-2}{n-2}, \quad (3)$$

where $0 \leq k \leq n$. Each irreducible tensor has $(2j+1)$ independent components. So that the total number of components in the reduction is $\sum_{j=0}^n (2j+1)N_n^{(j)} = 3^n$.

The natural projection of x^j onto the irreducible subspace H_j^j of traceless symmetric tensors of order j is denoted by $E^{(j)} = E_{k_1 k_2 \dots k_j; l_1 l_2 \dots l_j}^{(j)}$. The principal element in the reduction procedure is the mappings $Q_{i_1 i_2 \dots i_n; k_1 k_2 \dots k_j}^{(0;q)}$ of the minimal rank tensor subspace $H_{j,q}^j$ onto $H_{j,q}^n$, we have chosen the mappings $Q_{i_1 i_2 \dots i_n; k_1 k_2 \dots k_j}^{(0;q)}$ such that they are orthonormal and g_{pq} will be reduced to identity matrix, δ_{ij} , where g_{pq} is a symmetric matrix which was used and defined in Andrews and Ghoul (1982) through the relation

$$g_{pq} E_{k_1 k_2 \dots k_j; l_1 l_2 \dots l_j}^{(j)} = Q_{i_1 i_2 \dots i_n; k_1 k_2 \dots k_j}^{(0;p)} Q_{i_1 i_2 \dots i_n; l_1 l_2 \dots l_j}^{(0;q)}. \quad (4)$$

In this work, this relation is reduced to

$$\delta_{pq} E_{k_1 k_2 \dots k_j; l_1 l_2 \dots l_j}^{(j)} = Q_{i_1 i_2 \dots i_n; k_1 k_2 \dots k_j}^{(0;p)} Q_{i_1 i_2 \dots i_n; l_1 l_2 \dots l_j}^{(0;q)}. \quad (5)$$

The mappings $Q_{k_1 k_2 \dots k_j; i_1 i_2 \dots i_n}^{(0;p)}$ dual to $Q_{i_1 i_2 \dots i_n; k_1 k_2 \dots k_j}^{(0;p)}$ are defined by the relation

$$Q_{k_1 k_2 \dots k_j; i_1 i_2 \dots i_n}^{(0;p)} = Q_{i_1 i_2 \dots i_n; k_1 k_2 \dots k_j}^{(0;p)} \quad (6)$$

The dual mappings extract the natural forms $t_{\lambda_1 \lambda_2 \dots \lambda_j}^{(j;p)}$ from the tensor $T_{i_1 i_2 \dots i_n}$ as

$$t_{\lambda_1 \lambda_2 \dots \lambda_j}^{(j;p)} = Q_{\lambda_1 \lambda_2 \dots \lambda_j; i_1 i_2 \dots i_n}^{(0;p)} T_{i_1 i_2 \dots i_n} \quad (7)$$

These tensors can be embedded in the tensor space of order n through the mapping $T_{i_1 i_2 \dots i_n}^{(j;q)} = Q_{i_1 i_2 \dots i_n; k_1 k_2 \dots k_j}^{(0;q)} t_{k_1 k_2 \dots k_j}^{(j;p)}$, (8)

or

$$T_{i_1 i_2 \dots i_n}^{(j;q)} = Q_{i_1 i_2 \dots i_n; k_1 k_2 \dots k_j}^{(0;q)} Q_{k_1 k_2 \dots k_j; l_1 l_2 \dots l_n}^{(0;p)} T_{l_1 l_2 \dots l_n}, \quad (9)$$

Before giving the results of the reduction of elastic constant tensor, it is illustrated that the orthogonality condition is satisfied during the decomposition procedure. According to orthogonal

process found and given in Appendix and taking into account the elastic symmetries such as equation (2), elastic constant tensor is decomposed as

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(2;1)} + C_{ijkl}^{(2;2)} + C_{ijkl}^{(4;1)}, \quad (10)$$

where

$$C_{ijkl}^{(0;1)} = \frac{1}{9} \delta_{ij} \delta_{kl} C_{ppqq}, \quad (11)$$

$$C_{ijkl}^{(0;2)} = \frac{1}{90} (3\delta_{ik} \delta_{jl} + 3\delta_{il} \delta_{jk} - 2\delta_{ij} \delta_{kl}) (3C_{ppqq} - C_{ppqq}), \quad (12)$$

$$C_{ijkl}^{(2;1)} = \frac{1}{5} (\delta_{ik} C_{jplp} + \delta_{ik} C_{jplp} + \delta_{il} C_{jpkp} + \delta_{il} C_{ipkp}) - \frac{2}{15} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) C_{ppqq}, \quad (13)$$

$$C_{ijkl}^{(2;2)} = \frac{1}{7} \delta_{ij} (5C_{klpp} - 4C_{kplp}) + \frac{1}{7} \delta_{kl} (5C_{ijpp} - 4C_{ipjp}) - \frac{2}{35} \delta_{ik} (5C_{jlpp} - 4C_{jplp}) - \frac{2}{35} \delta_{jl} (5C_{ikpp} - 4C_{ipkp}) - \frac{2}{35} \delta_{il} (5C_{jkpp} - 4C_{jpkp}) - \frac{2}{35} \delta_{jk} (5C_{ilpp} - 4C_{iplp}) + \frac{2}{105} (2\delta_{jk} \delta_{il} + 2\delta_{ik} \delta_{jl} - 5\delta_{ij} \delta_{kl}) (5C_{ppqq} - 4C_{ppqq}), \quad (14)$$

$$C_{ijkl}^{(4;1)} = (C_{ijkl} + C_{iklj} + C_{iljk})/3 - [(C_{ijmm} + 2C_{imjm})\delta_{kl} + (C_{klmm} + 2C_{kmlm})\delta_{ij} + (C_{ikmm} + 2C_{imkm})\delta_{jl} + (C_{jlmm} + 2C_{jmlm})\delta_{ik} + (C_{ilmm} + 2C_{imlm})\delta_{jk} + (C_{jkmm} + C_{jmkm})\delta_{il}] / 21 + (C_{ppmm} + 2C_{pmpm})(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) / 105, \quad (15)$$

where $C_{ijkl}^{(0;1)}$, $C_{ijkl}^{(0;2)}$ are scalar parts, $C_{ijkl}^{(2;1)}$, $C_{ijkl}^{(2;2)}$ are deviators and $C_{ijkl}^{(4;1)}$ is the nonor part. As it is given by the group representation theory for the elastic constant tensor that $2D_0 + 2D_2 + D_4$ where the subscripts denote the weight of the representation, (See Schouten (1951) and Heine (1960)). This is the decomposition for triclinic materials which are anisotropic materials with no elastic symmetries.

2.1.1. For isotropic materials

There are only two irreducible parts for isotropic materials which are the scalar parts; $C_{ijkl}^{(0;1)}$ and $C_{ijkl}^{(0;2)}$ mentioned in equations (11) and (12). By rewriting these parts in matrix form, we get

$$C_{mn}^{(0;1)} = K \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_{mn}^{(0;2)} = \frac{G}{3} \begin{bmatrix} 4 & -2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad (16)$$

If we add those two parts, isotropic elastic constant tensor is obtained by irreducible decomposition method, isotropic elastic constant tensor can be rewritten instead of well known traditional form for isotropic materials in literature as

$$C_{ijkl} = K\delta_{ij}\delta_{kl} + 2G\left(\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \quad (17)$$

Where $K = (C_{11} + 2C_{12})/3$ and $G = \mu = (C_{11} - C_{12})/2$, where K is the bulk modulus and G is the shear modulus. In equation (16) the decomposed parts are orthogonal to each other whereas the decomposed parts in traditional form are not orthogonal. From equations (1) and (17), the stress-strain relation, for isotropic materials, can be obtained as

$$\sigma_{ij} = K\varepsilon_{rr}\delta_{ij} + 2G\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{rr}\delta_{ij}\right) \quad (18)$$

This is also different than the well known traditional form and it is a new form in stress-strain relations in the literature.

2.1.2. For cubic materials

For cubic symmetry, there are three irreducible parts which are two scalar parts; $C_{ijkl}^{(0;1)}$, $C_{ijkl}^{(0;2)}$ (given in equations (11) and (12)), and a nonor part $C_{ijkl}^{(4;1)}$ in equation (15). By arranging and rewriting these parts, we obtain

$$C_{ijkl}^{(0;1)} = \frac{1}{9}\delta_{ij}\delta_{kl}(C_{11} + 2C_{12}), \quad (19)$$

$$C_{ijkl}^{(0;2)} = \frac{1}{15}(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})(C_{11} - C_{12} + 3C_{44}), \quad (20)$$

$$C_{ijkl}^{(4;1)} = \frac{1}{5}(5\delta_{ij}\delta_{kl}\delta_{jk} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl})(C_{11} - C_{12} - 2C_{44}), \quad (21)$$

If we add equations (19), (20) and (21), elastic constant tensor for cubic material is obtained by irreducible decomposition method as

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(4;1)} \quad (22)$$

2.1.3. For transversely isotropic materials

There are five irreducible parts for transversely isotropic materials which are two scalar parts; $C_{ijkl}^{(0;1)}$, $C_{ijkl}^{(0;2)}$, two deviators; $C_{ijkl}^{(2;1)}$, $C_{ijkl}^{(2;2)}$, and a nonor part $C_{ijkl}^{(4;1)}$, mentioned in equations (11), (12); (13), (14) and (15) respectively. These parts are as follows:

$$C_{ijkl}^{(0;1)} = \frac{a}{9} \delta_{ij} \delta_{kl}, \quad (23)$$

$$C_{ijkl}^{(0;2)} = \frac{b}{90} (3\delta_{ik} \delta_{jl} + 3\delta_{il} \delta_{jk} - 2\delta_{ij} \delta_{kl}), \quad (24)$$

$$C_{ijkl}^{(2;1)} = \frac{c}{30} (2\delta_{ik} \delta_{jl} + 2\delta_{il} \delta_{jk} - 3\delta_{3i} \delta_{3k} \delta_{jl} - 3\delta_{3j} \delta_{3k} \delta_{il} + \delta_{3i} \delta_{3l} \delta_{jk} + \delta_{3j} \delta_{3l} \delta_{ik}), \quad (25)$$

$$C_{ijkl}^{(2;2)} = \frac{d}{105} (-4\delta_{ik} \delta_{jl} - 4\delta_{il} \delta_{jk} + 10\delta_{ij} \delta_{kl} - 15\delta_{3i} \delta_{3j} \delta_{kl} - 15\delta_{3k} \delta_{3l} \delta_{ij} + 6\delta_{3i} \delta_{3k} \delta_{jl} + 6\delta_{3j} \delta_{3l} \delta_{ik} + 6\delta_{3j} \delta_{3k} \delta_{il} + 6\delta_{3i} \delta_{3l} \delta_{jk}), \quad (26)$$

$$C_{ijkl}^{(4;1)} = \frac{e}{35} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} + \delta_{ij} \delta_{kl} - 5\delta_{3i} \delta_{3j} \delta_{kl} - 5\delta_{3k} \delta_{3l} \delta_{ij} - 5\delta_{3i} \delta_{3k} \delta_{jl} - 5\delta_{3j} \delta_{3k} \delta_{il} - 5\delta_{3i} \delta_{3l} \delta_{jk} - 5\delta_{3j} \delta_{3l} \delta_{ik} + 35\delta_{3i} \delta_{3j} \delta_{3k} \delta_{3l}). \quad (27)$$

Where $a = 2C_{11} + 2C_{12} + 4C_{13} + C_{33}$, $b = 7C_{11} - 5C_{12} + 2C_{33} + 12C_{44} - 4C_{13}$,

$c = 3C_{11} - 2C_{33} - 2C_{44} - C_{12}$, $d = 7C_{12} - C_{33} - C_{11} - 5C_{13} + 4C_{44}$ and

$e = C_{11} + C_{33} - 2C_{13} - 4C_{44}$.

If we add those five parts, elastic constant tensor for transversely isotropic material is obtained by irreducible decomposition method as

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(2;1)} + C_{ijkl}^{(2;2)} + C_{ijkl}^{(4;1)} \quad (28)$$

2.1.4. For tetragonal materials

For tetragonal materials, there are five irreducible parts which are two scalar parts; $C_{ijkl}^{(0;1)}$, $C_{ijkl}^{(0;2)}$, two deviators; $C_{ijkl}^{(2;1)}$, $C_{ijkl}^{(2;2)}$, and a nonor part $C_{ijkl}^{(4;1)}$ (given in equations (11), (12), (13), (14) and (15) respectively). These parts are arranged and rewritten to obtain the followings:

$$C_{ijkl}^{(0;1)} = \frac{a}{9} \delta_{ij} \delta_{kl}, \quad (29)$$

$$C_{ijkl}^{(0;2)} = \frac{b}{45} (3\delta_{ik} \delta_{jl} + 3\delta_{il} \delta_{jk} - 2\delta_{ij} \delta_{kl}), \quad (30)$$

$$C_{ijkl}^{(2;1)} = \frac{c}{15} (2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} - 3\delta_{3i}\delta_{3k}\delta_{jl} - 3\delta_{3j}\delta_{3k}\delta_{il} + \delta_{3i}\delta_{3l}\delta_{jk} + \delta_{3j}\delta_{3l}\delta_{ik}), \quad (31)$$

$$C_{ijkl}^{(2;2)} = \frac{d}{105} (-4\delta_{ik}\delta_{jl} - 4\delta_{il}\delta_{jk} + 10\delta_{ij}\delta_{kl} - 15\delta_{3i}\delta_{3j}\delta_{kl} - 15\delta_{3k}\delta_{3l}\delta_{ij} + 6\delta_{3i}\delta_{3k}\delta_{jl} + 6\delta_{3j}\delta_{3l}\delta_{ik} + 6\delta_{3j}\delta_{3k}\delta_{il} + 6\delta_{3i}\delta_{3l}\delta_{jk}), \quad (32)$$

$$C_{ijkl}^{(4;1)} = \frac{1}{35} (e(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl}) - (e+f)(\delta_{3i}\delta_{3j}\delta_{kl} + \delta_{3k}\delta_{3l}\delta_{ij}) - (e+f)(\delta_{3i}\delta_{3k}\delta_{jl} + \delta_{3j}\delta_{3k}\delta_{il} + \delta_{3i}\delta_{3l}\delta_{jk} + \delta_{3j}\delta_{3l}\delta_{ik}) + 35(C_{33} + C_{12} + 2C_{66} - 2C_{13} - 4C_{44})\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l} + 35(C_{11} - C_{12} - 2C_{66})(\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l})). \quad (33)$$

Where $a = 2C_{11} + 2C_{12} + 4C_{13} + C_{33}$, $b = 2C_{11} - C_{12} + C_{33} + 3C_{66} + 6C_{44} - 2C_{13}$,

$c = C_{11} - C_{33} - C_{44} + C_{66}$, $d = C_{11} - C_{33} + 5C_{12} - 5C_{13} + 4C_{44} - 4C_{66}$ and

$e = 9C_{12} + 18C_{66} - 8C_{11} - 2C_{13} - 4C_{44} + C_{33}$, $f = 4C_{33} + 3C_{11} - 8C_{13} + C_{12} - 16C_{44} + 2C_{66}$.

2.1.5. For trigonal materials

For trigonal materials, there are five irreducible parts which are two scalar parts; $C_{ijkl}^{(0;1)}$, $C_{ijkl}^{(0;2)}$, two deviators; $C_{ijkl}^{(2;1)}$, $C_{ijkl}^{(2;2)}$ and a nonor part $C_{ijkl}^{(4;1)}$ (given in equations (11), (12), (13), (14) and (15) respectively). These parts are arranged and then we obtain

$$C_{ijkl}^{(0;1)} = \frac{a}{9} \delta_{ij} \delta_{kl}, \quad C_{ijkl}^{(0;2)} = \frac{b}{90} (3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}), \quad (34)$$

$$C_{ijkl}^{(2;1)} = \frac{c}{30} (2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} - 3\delta_{3i}\delta_{3k}\delta_{jl} - 3\delta_{3j}\delta_{3k}\delta_{il} + \delta_{3i}\delta_{3l}\delta_{jk} + \delta_{3j}\delta_{3l}\delta_{ik}), \quad (35)$$

$$C_{ijkl}^{(2;2)} = \frac{d}{105} (-4\delta_{ik}\delta_{jl} - 4\delta_{il}\delta_{jk} + 10\delta_{ij}\delta_{kl} - 15\delta_{3i}\delta_{3j}\delta_{kl} - 15\delta_{3k}\delta_{3l}\delta_{ij} + 6\delta_{3i}\delta_{3k}\delta_{jl} + 6\delta_{3j}\delta_{3l}\delta_{ik} + 6\delta_{3j}\delta_{3k}\delta_{il} + 6\delta_{3i}\delta_{3l}\delta_{jk}), \quad (36)$$

$$C_{ijkl}^{(4;1)} = \frac{e}{35}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl} - 5\delta_{3i}\delta_{3j}\delta_{kl} - 5\delta_{3k}\delta_{3l}\delta_{ij} - 5\delta_{3i}\delta_{3k}\delta_{jl} - 5\delta_{3j}\delta_{3k}\delta_{il} - 5\delta_{3i}\delta_{3l}\delta_{jk} - 5\delta_{3j}\delta_{3l}\delta_{ik} + 35\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l}) + C_{14}(\delta_{1i}\delta_{1j}\delta_{2k}\delta_{3l} + \delta_{1i}\delta_{1j}\delta_{3k}\delta_{2l} + \delta_{3i}\delta_{2j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{3j}\delta_{1k}\delta_{1l} + \delta_{3i}\delta_{1j}\delta_{2k}\delta_{1l} + \delta_{3i}\delta_{1j}\delta_{1k}\delta_{2l} + \delta_{1i}\delta_{3j}\delta_{2k}\delta_{1l} + \delta_{1i}\delta_{3j}\delta_{1k}\delta_{2l} + \delta_{1i}\delta_{2j}\delta_{1k}\delta_{3l} + \delta_{1i}\delta_{2j}\delta_{3k}\delta_{1l} + \delta_{2i}\delta_{1j}\delta_{1k}\delta_{3l} + \delta_{2i}\delta_{1j}\delta_{3k}\delta_{1l} - \delta_{2i}\delta_{2j}\delta_{2k}\delta_{3l} - \delta_{2i}\delta_{2j}\delta_{3k}\delta_{2l} - \delta_{3i}\delta_{2j}\delta_{2k}\delta_{2l} - \delta_{2i}\delta_{3j}\delta_{2k}\delta_{2l}),$$

(37)

$$\text{where } a = 2C_{11} + 2C_{12} + 4C_{13} + C_{33}, \quad b = 7C_{11} - 5C_{12} + 2C_{33} + 12C_{44} - 4C_{13},$$

$$c = 3C_{11} - 2C_{33} - 2C_{44} - C_{12}, \quad d = 7C_{12} - C_{33} - C_{11} - 5C_{13} + 4C_{44} \quad \text{and}$$

$$e = C_{11} + C_{33} - 2C_{13} - 4C_{44}.$$

If we add those five parts, elastic constant tensor for trigonal material, represented by irreducible decomposition method as

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(2;1)} + C_{ijkl}^{(2;2)} + C_{ijkl}^{(4;1)} \quad (38)$$

2.1.6. For orthorhombic materials

There are five irreducible parts for orthorhombic materials which are two scalar parts; $C_{ijkl}^{(0;1)}$ in (11), $C_{ijkl}^{(0;2)}$ in (12), two deviators; $C_{ijkl}^{(2;1)}$ in (13), $C_{ijkl}^{(2;2)}$ in (14) and a nonor part $C_{ijkl}^{(4;1)}$ in (15). These parts are rewritten as

$$C_{ijkl}^{(0;1)} = \frac{a}{9}\delta_{ij}\delta_{kl}, \quad (39)$$

$$C_{ijkl}^{(0;2)} = \frac{b}{45}(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}), \quad (40)$$

$$C_{ijkl}^{(2;1)} = \frac{1}{15}(2j(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (2c + d)(\delta_{1i}\delta_{1k}\delta_{jl} + \delta_{1j}\delta_{1k}\delta_{il} + \delta_{1i}\delta_{1l}\delta_{jk} + \delta_{1j}\delta_{1l}\delta_{ik}) + (d - j)(\delta_{2i}\delta_{2k}\delta_{jl} + \delta_{2j}\delta_{2k}\delta_{il} + \delta_{2i}\delta_{2l}\delta_{jk} + \delta_{2j}\delta_{2l}\delta_{ik})), \quad (41)$$

$$C_{ijkl}^{(2;2)} = \frac{1}{105}(-4k(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + 10k\delta_{ij}\delta_{kl} - (4e + 2f)(\delta_{1i}\delta_{1k}\delta_{jl} + \delta_{1j}\delta_{1k}\delta_{il} + \delta_{1i}\delta_{1l}\delta_{jk} + \delta_{1j}\delta_{1l}\delta_{ik}) + (4k + 2e)(\delta_{2i}\delta_{2k}\delta_{jl} + \delta_{2j}\delta_{2k}\delta_{il} + \delta_{2i}\delta_{2l}\delta_{jk} + \delta_{2j}\delta_{2l}\delta_{ik}) + (5f - 5k)(\delta_{2k}\delta_{2l}\delta_{ij} + \delta_{2i}\delta_{2j}\delta_{kl}) + (5e - 5k)(\delta_{1k}\delta_{1l}\delta_{ij} + \delta_{1i}\delta_{1j}\delta_{kl})), \quad (42)$$

$$\begin{aligned}
 C_{ijkl}^{(4;1)} = & \frac{1}{35}(-(n+i)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (3n+2i)\delta_{ij}\delta_{kl} + (i-g)(\delta_{li}\delta_{lk}\delta_{jl} + \delta_{lj}\delta_{lk}\delta_{il} + \delta_{li}\delta_{lk}\delta_{jk} + \delta_{lj}\delta_{lk}\delta_{ik}) + \\
 & (i-h)(\delta_{2i}\delta_{2k}\delta_{jl} + \delta_{2j}\delta_{2k}\delta_{il} + \delta_{2i}\delta_{2l}\delta_{jk} + \delta_{2j}\delta_{2l}\delta_{ik}) + \\
 & (2m-3n+3g)(\delta_{2k}\delta_{2l}\delta_{ij} + \delta_{2i}\delta_{2j}\delta_{kl}) + (2l-3n+3h)(\delta_{lk}\delta_{li}\delta_{ij} + \delta_{li}\delta_{lj}\delta_{kl}) + \\
 & 35(C_{12} + C_{33} + 2C_{66} - C_{23} - C_{13} - 2C_{44} - 2C_{55})(\delta_{li}\delta_{lj}\delta_{2k}\delta_{2l} + \delta_{2i}\delta_{2j}\delta_{lk}\delta_{li}) + \\
 & 35(C_{11} + C_{33} - 2C_{13} - 4C_{55})\delta_{li}\delta_{lj}\delta_{lk}\delta_{li} + 35(C_{33} + C_{22} - 2C_{23} - 4C_{44})\delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}. \quad (43)
 \end{aligned}$$

where $a = C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23}$,

$$b = C_{11} + C_{22} + C_{33} - C_{12} - C_{13} - C_{23} + 3C_{44} + 3C_{55} + 3C_{66},$$

$$c = 2C_{11} - C_{22} - C_{33} - 2C_{44} + C_{55} + C_{66},$$

$$d = 2C_{22} - C_{11} - C_{33} + C_{44} - 2C_{55} + C_{66},$$

$$e = 2C_{11} - C_{22} - C_{33} + 5C_{12} + 5C_{13} - 10C_{23} + 8C_{44} - 4C_{55} - 4C_{66},$$

$$f = 2C_{22} - C_{11} - C_{33} + 5C_{12} - 10C_{13} + 5C_{23} - 4C_{44} + 8C_{55} - 4C_{66},$$

$$g = C_{11} - 4C_{22} - 4C_{33} - C_{12} - C_{13} + 9C_{23} + 18C_{44} - 2C_{55} - 2C_{66},$$

$$h = C_{22} - 4C_{11} - 4C_{33} - C_{12} + 9C_{13} - C_{23} - 2C_{44} + 18C_{55} - 2C_{66},$$

$$i = C_{33} - 4C_{11} - 4C_{22} + 9C_{12} - C_{13} - C_{23} - 2C_{44} - 2C_{55} + 18C_{66},$$

$$j = -(c+d), \quad k = -(e+f), \quad l = -(h+i), \quad m = -(j+i),$$

$$n = -(j+h).$$

If we add those five parts, elastic constant tensor for orthorhombic material, obtained by irreducible decomposition method as

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(2;1)} + C_{ijkl}^{(2;2)} + C_{ijkl}^{(4;1)} \quad (44)$$

2.2. By harmonic decomposition method

In harmonic decomposition, the action of $SO(3)$ on a vector space is said to be irreducible when there are no proper invariant subspaces. It is deduced that there is a decomposition of the space of elastic constant tensors ($\mathbf{E}$ la) into a direct sum of orthogonal subspaces on which the action of $SO(3)$ is irreducible. An important theorem of group representation theory can be summarized as: every space on which the group of rotations acts irreducibly is isomorphic through an $SO(3)$ -invariant map with an appropriate space of harmonic polynomials. In view of isomorphism, there is a decomposition of $\mathbf{E}$ la into a direct sum of spaces of harmonic tensors. (See, for instance; Forte and Vianello (1996)) Besides, there is an $SO(3)$ -invariant isomorphism between $\mathbf{E}$ la and the direct sum $\mathbf{R} \oplus \mathbf{R} \oplus \mathbf{Dev} \oplus \mathbf{Dev} \oplus \mathbf{H}$ rm. We give a brief review for this method as follows: The decomposition of elastic constant tensor for anisotropic materials possessing triclinic symmetry, we obtain

$$C_{ijkl} = H_{ijkl} + [\delta_{ij}H_{kl} + \delta_{kl}H_{ij} + \delta_{ik}H_{jl} + \delta_{jl}H_{ik} + \delta_{il}H_{jk} + \delta_{jk}H_{il}] + H[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] + \delta_{ij}h_{kl} + \delta_{kl}h_{ij} - \frac{1}{2}\delta_{ik}h_{jl} - \frac{1}{2}\delta_{jl}h_{ik} - \frac{1}{2}\delta_{il}h_{jk} - \frac{1}{2}\delta_{jk}h_{il} + h(\delta_{ij}\delta_{kl} - \frac{1}{2}\delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{il}\delta_{jk}). \quad (45)$$

Where $H = \frac{1}{45}(C_{ppqq} + 2C_{pqpp})$, $h = \frac{1}{9}(C_{ppqq} - C_{pqpp})$,

$$H_{ij} = \frac{1}{21}\left[C_{ijkk} - \frac{1}{3}C_{jjkk}\delta_{ij} + 2\left(C_{ikjk} - \frac{1}{3}C_{jkjk}\delta_{ij}\right)\right], \quad h_{ij} = \frac{2}{3}(C_{ijpp} - C_{ipjp}) - \frac{2}{9}\delta_{ij}(C_{rrpp} - C_{rprp}).$$

In this method, the notation of Baerheim (1993) has been used.

The total scalar (isotropic) part of harmonic decomposition (obtained from equation (45)) is (denoted as S)

$$S = \frac{\delta_{ij}\delta_{kl}}{15}(2C_{ppqq} - C_{pqpp}) + \frac{(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}{30}(3C_{pqpp} - C_{ppqq}). \quad (46)$$

Furthermore the total deviatoric part or second rank traceless tensor is composed of summation of the linear combination of second order tensors (H_{ij} and h_{ij}) given in equation (45), which is (denoted as D)

$$D = \frac{1}{7}\delta_{kl}(5C_{ijpp} - 4C_{ipjp}) + \frac{1}{7}\delta_{ij}(5C_{klpp} - 4C_{kplp}) + \frac{1}{7}\delta_{ik}(3C_{jplp} - 2C_{jlpp}) + \frac{1}{7}\delta_{il}(3C_{jpkp} - 2C_{jkpp}) + \frac{1}{7}\delta_{jl}(3C_{ipkp} - 2C_{ikpp}) + \frac{1}{7}\delta_{jk}(3C_{iplp} - 2C_{ilpp}) + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\left(\frac{4}{21}C_{pprr} - \frac{2}{7}C_{rprp}\right) + \delta_{ij}\delta_{kl}\left(\frac{8}{21}C_{rprp} - \frac{10}{21}C_{pprr}\right). \quad (47)$$

From equation (45), harmonic (nonor) part is obtained as the same expression as $C_{ijkl}^{(4;1)}$ given in equation (15).

Moreover, the results for elastic constant tensor decomposition are given by Onat (1984, 1994) and quoted by Cowin (1989), Forte and Vianello (1996, 2006), Ting (2006), Zheng (2007), Annin and Ostrosablin (2008) in which elastic constant tensor is decomposed into two scalar, two deviatoric and one nonor parts. These decompositions are the same as harmonic decomposition method since scalar, deviatoric and nonor parts are common and they are identical with those obtained from harmonic decomposition method, only difference here is notations used for scalar, traceless symmetric second rank tensors and nonor parts. According to these studies, decomposition of elastic constant tensor for anisotropic materials possessing triclinic symmetry is expressed as follows:

$$C_{ijkl} = \frac{1}{15}(2C_{ppqq} - C_{pqpp})\delta_{ij}\delta_{kl} + \frac{1}{30}(3C_{pqpp} - C_{ppqq})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \delta_{ij}A_{kl} + \delta_{kl}A_{ij} + \delta_{ik}B_{jl} + \delta_{jl}B_{ik} + \delta_{il}B_{jk} + \delta_{jk}B_{il} + Z_{ijkl}, \quad (48)$$

From equation (48), total scalar part is

$$\frac{1}{15}(2C_{ppqq} - C_{ppqq})\delta_{ij}\delta_{kl} + \frac{1}{30}(3C_{ppqq} - C_{ppqq})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (49)$$

The total deviatoric part is

$$\delta_{ij}A_{kl} + \delta_{kl}A_{ij} + \delta_{ik}B_{jl} + \delta_{jl}B_{ik} + \delta_{il}B_{jk} + \delta_{jk}B_{il}. \quad (50)$$

The components of deviatoric part are

$$A_{ij} = (15C_{ijkk} - 12C_{ikjk} - 5\delta_{ij}C_{ppkk} + 4\delta_{ij}C_{pkpk})/21, \quad (51)$$

$$B_{ij} = (-6C_{ijkk} + 9C_{ikjk} + 2\delta_{ij}C_{ppkk} - 3\delta_{ij}C_{pkpk})/21. \quad (52)$$

Finally nonor part is the same as $C_{ijkl}^{(4;1)}$ given in equation (15). Furthermore the decomposition of elastic constant tensor given in Forte and Vianello (1996, 2006) contain misprints in components of scalar part and total deviatoric part. In equations (49) and (50), these parts are corrected.

2.2.1. For isotropic materials

Like irreducible decomposition method, there are two irreducible parts constituted total isotropic (scalar) parts which are found by using eqs. (45) and (46) and these decomposed parts are

$$\frac{1}{15}(2C_{ppqq} - C_{ppqq})\delta_{ij}\delta_{kl}, \quad \frac{1}{30}(3C_{ppqq} - C_{ppqq})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (53)$$

The sum of the parts in equation (53) gives total scalar part which is obtained in equation (46).

2.2.2. For cubic materials

There exists three decomposed parts which are two scalars and a harmonic part. The decomposed parts of total scalar part are identical with those given in equation (53). Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in equation (21). If we add those parts, elastic constant tensor for cubic materials is obtained by harmonic decomposition method.

2.2.3. For transversely isotropic materials

There are five irreducible parts for transversely isotropic materials which are two scalars and two deviators and a nonor (harmonic) part. The decomposed parts of total scalar part are the same as those given in equation (53) and decomposed parts of total deviatoric part which are $C_{ijkl}^{(2;1)}$ and $C_{ijkl}^{(2;2)}$ found in equations (25) and (26) respectively, are obtained by arranging equation (47) so we get total deviatoric part in harmonic decomposition method. Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in equation (27), we obtain harmonic part for transversely isotropic materials in harmonic decomposition method.

2.2.4. For tetragonal materials

By applying harmonic decomposition method, elastic constant tensor is decomposed into the parts namely as two scalar, two deviatoric and nonor. Decomposed parts are the same as those given in equation (53). Parts of total deviatoric part which are $C_{ijkl}^{(2;1)}$ and $C_{ijkl}^{(2;2)}$ found in equations (31) and

(32) respectively, are obtained by arranging equation (47) so we get total deviatoric part in harmonic decomposition method. Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in equation (33), we obtain harmonic part for tetragonal materials in harmonic decomposition method.

2.2.5. For trigonal materials

For trigonal materials, five decomposed parts, which are two scalars, two deviatoric and nonor, are obtained by harmonic decomposition method. Decomposed parts of total scalar part are identical with the expressions given in equation (53). Terms of total deviatoric parts can be obtained by rearranging equation (47) so we get these parts which are the same as those given in equations (35) and (36). Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in equation (37), we obtain harmonic part for trigonal materials in harmonic decomposition method.

2.2.6. For orthorhombic materials

There are five irreducible parts constituted total scalar, total deviatoric and nonor parts. Total scalar part is composed of the parts given in (53). By rearranging equation (47) so we get two parts forming total deviatoric part which are the same as those given in equations (41) and (42). Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in equation (43), we obtain harmonic part for orthorhombic materials in harmonic decomposition method. So elastic constant tensor for orthorhombic materials can be represented by summing equations (53), (41), (42) and (43) in harmonic decomposition method.

3. NUMERICAL EXAMPLES

The decomposition methods can be carried out for various anisotropic materials such that reactor pressure vessel steel, aluminium antimonide, polysytrene, zircon, haematite and olivine are selected as example materials for isotropic, cubic, transversely isotropic, tetragonal, trigonal and orthorhombic symmetries (for detail description of these symmetries, see Nye (1957), Mehrabadi and Cowin (1995)) respectively. The irreducible parts in matrix form are also given. Elastic constant data for AlSb, Polysytrene, Zircon and Haematite are taken from Landolt-Börnstein (1979). Decomposition processes, irreducible and harmonic decomposition are performed by simple computer programs written in MATLAB in order to ease the calculations. All units are in Gpa (10^{10} dyn · cm⁻²) .

3.1. Reactor pressure vessel steel (RPV steel) (Data taken from Cheong (1999))

$$C_{ij} = \begin{bmatrix} 277.001 & 118.715 & 118.715 & 0 & 0 & 0 \\ 118.715 & 277.001 & 118.715 & 0 & 0 & 0 \\ 118.715 & 118.715 & 277.001 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix} \quad (54)$$

3.1.1. By irreducible decomposition method

$$C_{mn}^{(0;1)} = \begin{bmatrix} 171.5 & 171.5 & 171.5 & 0 & 0 & 0 \\ 171.5 & 171.5 & 171.5 & 0 & 0 & 0 \\ 171.5 & 171.5 & 171.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C_{mn}^{(0;2)} = \begin{bmatrix} 105.524 & -52.762 & -52.762 & 0 & 0 & 0 \\ -52.762 & 105.524 & -52.762 & 0 & 0 & 0 \\ -52.762 & -52.762 & 105.524 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix} \quad (55)$$

3.1.2. By harmonic decomposition method

$$S_1 = \begin{bmatrix} 118.715 & 118.715 & 118.715 & 0 & 0 & 0 \\ 118.715 & 118.715 & 118.715 & 0 & 0 & 0 \\ 118.715 & 118.715 & 118.715 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 158.286 & 0 & 0 & 0 & 0 & 0 \\ 0 & 158.286 & 0 & 0 & 0 & 0 \\ 0 & 0 & 158.286 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix} \quad (56)$$

3.2. Aluminium Antimonide (AlSb)

$$C_{ij} = \begin{bmatrix} 87.7 & 43.4 & 43.4 & 0 & 0 & 0 \\ 43.4 & 87.7 & 43.4 & 0 & 0 & 0 \\ 43.4 & 43.4 & 87.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 40.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 40.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 40.8 \end{bmatrix} \quad (57)$$

3.2.1. By irreducible decomposition method

$$C_{mn}^{(0;1)} = \begin{bmatrix} 58.2 & 58.2 & 58.2 & 0 & 0 & 0 \\ 58.2 & 58.2 & 58.2 & 0 & 0 & 0 \\ 58.2 & 58.2 & 58.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C_{mn}^{(0;2)} = \begin{bmatrix} 44.4 & -22.2 & -22.2 & 0 & 0 & 0 \\ -22.2 & 44.4 & -22.2 & 0 & 0 & 0 \\ -22.2 & -22.2 & 44.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 33.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 33.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 33.3 \end{bmatrix}, \quad (58)$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} -14.9 & 7.5 & 7.5 & 0 & 0 & 0 \\ 7.5 & -14.9 & 7.5 & 0 & 0 & 0 \\ 7.5 & 7.5 & -14.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.5 \end{bmatrix}. \quad (59)$$

3.2.2. By harmonic decomposition method

$$S_1 = \begin{bmatrix} 35.9 & 35.9 & 35.9 & 0 & 0 & 0 \\ 35.9 & 35.9 & 35.9 & 0 & 0 & 0 \\ 35.9 & 35.9 & 35.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 66.68 & 0 & 0 & 0 & 0 & 0 \\ 0 & 66.68 & 0 & 0 & 0 & 0 \\ 0 & 0 & 66.68 & 0 & 0 & 0 \\ 0 & 0 & 0 & 33.34 & 0 & 0 \\ 0 & 0 & 0 & 0 & 33.34 & 0 \\ 0 & 0 & 0 & 0 & 0 & 33.34 \end{bmatrix}, \quad (60)$$

$$H = \begin{bmatrix} -14.9 & 7.5 & 7.5 & 0 & 0 & 0 \\ 7.5 & -14.9 & 7.5 & 0 & 0 & 0 \\ 7.5 & 7.5 & -14.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.5 \end{bmatrix}. \quad (61)$$

3.3. Polystyrene

$$C_{ij} = \begin{bmatrix} 5.20 & 2.75 & 2.75 & 0 & 0 & 0 \\ 2.75 & 5.20 & 2.75 & 0 & 0 & 0 \\ 2.75 & 2.75 & 5.70 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.225 \end{bmatrix} \quad (62)$$

3.3.1. By irreducible decomposition method

$$C_{mn}^{(0;1)} = \begin{bmatrix} 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0 \\ 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0 \\ 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C_{mn}^{(0;2)} = \begin{bmatrix} 1.7178 & -0.8589 & -0.8589 & 0 & 0 & 0 \\ -0.8589 & 1.7178 & -0.8589 & 0 & 0 & 0 \\ -0.8589 & -0.8589 & 1.7178 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.2883 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.2883 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.2883 \end{bmatrix}, \quad (63)$$

$$C_{mn}^{(2;1)} = \begin{bmatrix} -0.1533 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1533 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3067 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0383 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0383 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0767 \end{bmatrix}, C_{mn}^{(2;2)} = \begin{bmatrix} -0.0038 & -0.0190 & 0.0095 & 0 & 0 & 0 \\ -0.0190 & -0.0038 & 0.0095 & 0 & 0 & 0 \\ 0.0095 & 0.0095 & 0.0076 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0038 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0038 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0076 \end{bmatrix}, \quad (64)$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} 0.0171 & 0.0057 & -0.0229 & 0 & 0 & 0 \\ 0.0057 & 0.0171 & -0.0229 & 0 & 0 & 0 \\ -0.0229 & -0.0229 & 0.0457 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0229 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0229 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0057 \end{bmatrix}, \quad (65)$$

3.3.2. By harmonic decomposition method

$$S_1 = \begin{bmatrix} 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0 \\ 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0 \\ 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 2.5767 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.5767 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.5767 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.2883 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.2883 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.2883 \end{bmatrix}, \quad (66)$$

$$D_1 = \begin{bmatrix} -0.1533 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1533 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3067 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0383 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0383 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0767 \end{bmatrix}, D_2 = \begin{bmatrix} -0.0038 & -0.0190 & 0.0095 & 0 & 0 & 0 \\ -0.0190 & -0.0038 & 0.0095 & 0 & 0 & 0 \\ 0.0095 & 0.0095 & 0.0076 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0038 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0038 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0076 \end{bmatrix}, \quad (67)$$

$$H = \begin{bmatrix} 0.0171 & 0.0057 & -0.0229 & 0 & 0 & 0 \\ 0.0057 & 0.0171 & -0.0229 & 0 & 0 & 0 \\ -0.0229 & -0.0229 & 0.0457 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0229 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0229 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0057 \end{bmatrix}, \quad (68)$$

3.4. Zircon ($ZrSiO_4$, metamict)

$$C_{ij} = \begin{bmatrix} 284 & 73 & 119 & 0 & 0 & 0 \\ 73 & 284 & 119 & 0 & 0 & 0 \\ 119 & 119 & 309 & 0 & 0 & 0 \\ 0 & 0 & 0 & 77.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 77.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 47.7 \end{bmatrix} \quad (69)$$

3.4.1. By irreducible decomposition method

$$C_{mn}^{(0;1)} = \begin{bmatrix} 166.56 & 166.56 & 166.56 & 0 & 0 & 0 \\ 166.56 & 166.56 & 166.56 & 0 & 0 & 0 \\ 166.56 & 166.56 & 166.56 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_{mn}^{(0;2)} = \begin{bmatrix} 104.36 & -52.18 & -52.18 & 0 & 0 & 0 \\ -52.18 & 104.36 & -52.18 & 0 & 0 & 0 \\ -52.18 & -52.18 & 104.36 & 0 & 0 & 0 \\ 0 & 0 & 0 & 78.27 & 0 & 0 \\ 0 & 0 & 0 & 0 & 78.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 78.27 \end{bmatrix}, \quad (70)$$

$$C_{mn}^{(2;1)} = \begin{bmatrix} -14.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -14.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 29.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.65 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.65 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.31 \end{bmatrix}, \quad C_{mn}^{(2;2)} = \begin{bmatrix} -2.59 & -12.9 & 6.47 & 0 & 0 & 0 \\ -12.9 & -2.59 & 6.47 & 0 & 0 & 0 \\ 6.47 & 6.47 & 5.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.59 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.59 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.17 \end{bmatrix}, \quad (71)$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} 16.657 & -13.11 & -3.54 & 0 & 0 & 0 \\ -13.11 & 16.657 & -3.54 & 0 & 0 & 0 \\ -3.54 & -3.54 & 7.09 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & -13.11 \end{bmatrix}. \quad (72)$$

3.4.2. By harmonic decomposition method

$$S_1 = \begin{bmatrix} 114.37 & 114.37 & 114.37 & 0 & 0 & 0 \\ 114.37 & 114.37 & 114.37 & 0 & 0 & 0 \\ 114.37 & 114.37 & 114.37 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 156.55 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156.55 & 0 & 0 & 0 & 0 \\ 0 & 0 & 156.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 78.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 78.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 78.3 \end{bmatrix}, \quad (73)$$

$$D_1 = \begin{bmatrix} -14.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -14.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 29.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.65 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.65 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.31 \end{bmatrix}, D_2 = \begin{bmatrix} -2.59 & -12.9 & 6.47 & 0 & 0 & 0 \\ -12.9 & -2.59 & 6.47 & 0 & 0 & 0 \\ 6.47 & 6.47 & 5.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.59 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.59 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.17 \end{bmatrix}, \quad (74)$$

$$H = \begin{bmatrix} 16.657 & -13.11 & -3.54 & 0 & 0 & 0 \\ -13.11 & 16.657 & -3.54 & 0 & 0 & 0 \\ -3.54 & -3.54 & 7.09 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & -13.11 \end{bmatrix}. \quad (75)$$

3.5. Haematite (Fe_2O_3)

$$C_{ij} = \begin{bmatrix} 242 & 54.9 & 15.7 & -12.7 & 0 & 0 \\ 54.9 & 242 & 15.7 & 12.7 & 0 & 0 \\ 15.7 & 15.7 & 228 & 0 & 0 & 0 \\ -12.7 & 12.7 & 0 & 85.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 85.3 & -12.7 \\ 0 & 0 & 0 & 0 & -12.7 & 93.55 \end{bmatrix} \quad (76)$$

3.5.1. By irreducible decomposition method

$$C_{mn}^{(0;1)} = \begin{bmatrix} 98.29 & 98.29 & 98.29 & 0 & 0 & 0 \\ 98.29 & 98.29 & 98.29 & 0 & 0 & 0 \\ 98.29 & 98.29 & 98.29 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C_{mn}^{(0;2)} = \begin{bmatrix} 126.06 & -63.03 & -63.03 & 0 & 0 & 0 \\ -63.03 & 126.06 & -63.03 & 0 & 0 & 0 \\ -63.03 & -63.03 & 126.06 & 0 & 0 & 0 \\ 0 & 0 & 0 & 94.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & 94.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & 94.54 \end{bmatrix}, \quad (77)$$

$$C_{mn}^{(2;1)} = \begin{bmatrix} 5.93 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.93 & 0 & 0 & 0 & 0 \\ 0 & 0 & -11.87 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.48 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.48 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.97 \end{bmatrix}, C_{mn}^{(2;2)} = \begin{bmatrix} 3.37 & 16.857 & -8.43 & 0 & 0 & 0 \\ 16.857 & 3.37 & -8.43 & 0 & 0 & 0 \\ -8.43 & -8.43 & -6.74 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.37 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.37 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6.74 \end{bmatrix}, \quad (78)$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} 8.35 & 2.783 & -11.13 & -12.7 & 0 & 0 \\ 2.783 & 8.35 & -11.13 & 12.7 & 0 & 0 \\ -11.13 & -11.13 & 22.26 & 0 & 0 & 0 \\ -12.7 & 12.7 & 0 & -11.13 & 0 & 0 \\ 0 & 0 & 0 & 0 & -11.13 & -12.7 \\ 0 & 0 & 0 & 0 & -12.7 & 2.783 \end{bmatrix} \quad (79)$$

3.5.2. By harmonic decomposition method

$$S_1 = \begin{bmatrix} 35.26 & 35.26 & 35.26 & 0 & 0 & 0 \\ 35.26 & 35.26 & 35.26 & 0 & 0 & 0 \\ 35.26 & 35.26 & 35.26 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 189.09 & 0 & 0 & 0 & 0 & 0 \\ 0 & 189.09 & 0 & 0 & 0 & 0 \\ 0 & 0 & 189.09 & 0 & 0 & 0 \\ 0 & 0 & 0 & 94.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & 94.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & 94.54 \end{bmatrix} \quad (80)$$

$$D_1 = \begin{bmatrix} 5.93 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.93 & 0 & 0 & 0 & 0 \\ 0 & 0 & -11.87 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.48 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.48 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.97 \end{bmatrix}, D_2 = \begin{bmatrix} 3.37 & 16.857 & -8.43 & 0 & 0 & 0 \\ 16.857 & 3.37 & -8.43 & 0 & 0 & 0 \\ -8.43 & -8.43 & -6.74 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.37 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.37 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6.74 \end{bmatrix}, \quad (81)$$

$$H = \begin{bmatrix} 8.35 & 2.783 & -11.13 & -12.7 & 0 & 0 \\ 2.783 & 8.35 & -11.13 & 12.7 & 0 & 0 \\ -11.13 & -11.13 & 22.26 & 0 & 0 & 0 \\ -12.7 & 12.7 & 0 & -11.13 & 0 & 0 \\ 0 & 0 & 0 & 0 & -11.13 & -12.7 \\ 0 & 0 & 0 & 0 & -12.7 & 2.783 \end{bmatrix}. \quad (82)$$

3.6. Olivine (Data taken from Chevrot, Browaeys (2004))

$$C_{ij} = \begin{bmatrix} 192 & 66 & 60 & 0 & 0 & 0 \\ 66 & 160 & 56 & 0 & 0 & 0 \\ 60 & 56 & 272 & 0 & 0 & 0 \\ 0 & 0 & 0 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 62 & 0 \\ 0 & 0 & 0 & 0 & 0 & 49 \end{bmatrix} \quad (83)$$

3.6.1. By irreducible decomposition method

$$C_{mn}^{(0;1)} = \begin{bmatrix} 109.78 & 109.78 & 109.78 & 0 & 0 & 0 \\ 109.78 & 109.78 & 109.78 & 0 & 0 & 0 \\ 109.78 & 109.78 & 109.78 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C_{mn}^{(0;2)} = \begin{bmatrix} 84.89 & -42.44 & -42.44 & 0 & 0 & 0 \\ -42.44 & 84.89 & -42.44 & 0 & 0 & 0 \\ -42.44 & -42.44 & 84.89 & 0 & 0 & 0 \\ 0 & 0 & 0 & 63.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 63.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 63.67 \end{bmatrix}, \quad (84)$$

$$C_{mn}^{(2;1)} = \begin{bmatrix} -15.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -42.40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 57.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -14.4 \end{bmatrix}, C_{mn}^{(2;2)} = \begin{bmatrix} 1.10 & -0.762 & 3.52 & 0 & 0 & 0 \\ -0.762 & -1.41 & -2.76 & 0 & 0 & 0 \\ 3.52 & -2.76 & 0.305 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.10 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.41 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.305 \end{bmatrix}, \quad (85)$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} 11.43 & -0.57 & -10.857 & 0 & 0 & 0 \\ -0.57 & 9.14 & -8.57 & 0 & 0 & 0 \\ -10.857 & -8.57 & 19.43 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8.57 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.857 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.57 \end{bmatrix} \quad (86)$$

3.6.2. By harmonic decomposition method

$$S_1 = \begin{bmatrix} 67.33 & 67.33 & 67.33 & 0 & 0 & 0 \\ 67.33 & 67.33 & 67.33 & 0 & 0 & 0 \\ 67.33 & 67.33 & 67.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 127.33 & 0 & 0 & 0 & 0 & 0 \\ 0 & 127.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 127.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 63.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 63.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 63.67 \end{bmatrix}, \quad (87)$$

$$D_1 = \begin{bmatrix} -15.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -42.40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 57.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -14.4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1.10 & -0.762 & 3.52 & 0 & 0 & 0 \\ -0.762 & -1.41 & -2.76 & 0 & 0 & 0 \\ 3.52 & -2.76 & 0.305 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.10 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.41 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.305 \end{bmatrix}, \quad (88)$$

$$H = \begin{bmatrix} 11.43 & -0.57 & -10.857 & 0 & 0 & 0 \\ -0.57 & 9.14 & -8.57 & 0 & 0 & 0 \\ -10.857 & -8.57 & 19.43 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8.57 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.857 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.57 \end{bmatrix} \quad (89)$$

4. DISCUSSION AND CONCLUSIONS

The decomposition methods are compared to emphasize the significant effect of these methods on elastic constant tensor by regarding the results of numerical examples. As a result critical relationships between them are found out. The first one is, like new method developed here, two scalars, two deviators and the nonor part are obtained in harmonic decomposition method. Hence total scalar, total deviatoric parts and nonor part are identical in two methods. In new irreducible method, components of total scalar parts; $C_{mn}^{(0;1)}$ and $C_{mn}^{(0;2)}$ are orthogonal to each other. Contrary to it, decomposed parts of total scalar part are not orthogonal to each other in harmonic decomposition method due to the expression for decomposition of elastic constant tensor given in equations (45) and (48). Since $\delta_{ij}\delta_{kl}$ is not orthogonal to $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ (denoted as $2I_{ijkl}$). If $\delta_{ij}\delta_{kl}$ and $2I_{ijkl}$ is replaced by the hydrostatic and deviatoric operators

$$I_{ijkl}^h = \frac{1}{3}\delta_{ij}\delta_{kl}, \quad I_{ijkl}^d = I_{ijkl} - I_{ijkl}^h, \quad (90)$$

respectively, then new method is obtained back in which decomposed parts of total scalar parts are orthogonal to each other. So the components of total scalar part in harmonic decomposition method

can take the form of $C_{mn}^{(0;1)}$ and $C_{mn}^{(0;2)}$. This case is a significant innovation for both decomposition methods for elastic constant tensor. It is the second relationship between both decomposition methods.

Besides, in irreducible decomposition method, by using equation (18), we obtain two physically meaningful parts which are

$$J_1 = 3KI_1, \quad (91)$$

and

$$(\sigma_{ij} - \frac{1}{3}\sigma_{rr}\delta_{ij}) = 2G(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{rr}\delta_{ij}), \quad (92)$$

where $J_1 = \sigma_{ii}$ and $I_1 = \varepsilon_{ii}$ are the first fundamental invariants of stress and strain tensors, respectively.

Equation (91) represents volume-change without distortion under hydrostatic stress and equation (92) represents shape-change without volume-change under deviatoric stress. So irreducible decomposition method gives two isotropic (scalar) parts which have significant physical meanings.

One of the important contribution of this work is that components of the scalar part in irreducible method are not equal to that in harmonic decomposition method, so it proves that there is not a unique decomposition for the scalar part, in other words total scalar part can be decomposed into infinitely many independent components. This case also indicates that total scalar, deviatoric and nonor parts of elastic constant tensor obtained from irreducible decomposition methods are the same as those of harmonic decomposition method.

Irreducible decomposition methods have many applications in different subjects of physics and engineering (atomic and molecular physics and the physics of condensed matter). In the mechanics of continuous media (for instance in elasticity studies, the strain and the stress tensors are decomposed into spherical and deviatoric parts), the hydrostatic pressure (spherical part) is connected to the change of volume without change of shape through the bulk modulus whereas the change of shape is connected to the deviatoric part of the stress through the shear modulus. In chemistry (see for instance; Jerphagnon et al. (1978)) (for example in the calculation of the rotational averages which arise in the theory of spectroscopic radiation molecule interactions in fluid media), based upon the principles of irreducible cartesian tensor analysis, the rotational averages cast in terms of irreducible tensor products are considerably simpler in structure than the corresponding results expressed in reducible form.

The decomposition methods of elastic constant tensor are applied to different fields of science and engineering. For instance, Geophysicists have used it in geophysical applications. (See Chevrot and Browaeys, (2004)) As an application for harmonic decomposition, it is possible to decide which type of symmetry a material has when the elastic constants are measured relative to an arbitrary coordinate system. A second rank symmetric tensor associated to the elastic constant tensor can be used to verify if the coordinate axes are the symmetry axes of the material and determine a symmetry coordinate system. (See Baerheim, (1993)) So comprehending the irreducible decomposition methods is considerable to understand the idea behind these decomposition methods as well as the physical properties of anisotropic materials.

Decomposition of the elastic constant tensor into irreducible orthogonal parts, offer not only valuable insight into the tensor structure but also simplify immensely the calculations of sums, products, inverses and inner products. It also provides a deeper understanding about elastic and mechanical behavior of anisotropic materials. It also has more significant effects on many applications in different

fields such as comparing the anisotropic properties of materials, examining the material symmetry types in detail, determination of materials possessing same crystal symmetry type which are highly anisotropic or close to isotropy.

APPENDIX

For irreducible decomposition method; the orthogonality condition is checked by using orthonormalized basis elements, the procedure is shown as

For fourth rank tensor $n = 4$ and $j = 0$, q will take the values 1, 2 and 3. For $q = 1$,

$$Q_{ijkl}^{(0;1)} = a \delta_{ij} \delta_{kl}. \quad (93)$$

Applying the normalization condition, equation (93) takes the following form:

$$Q_{ijkl}^{(0;1)} Q_{ijkl}^{(0;1)} = a^2 \delta_{ij} \delta_{kl} \delta_{ij} \delta_{kl} = 1 \quad (94)$$

$a = \frac{1}{3}$, So the first idempotent is

$$Q_{ijkl}^{(0;1)} = \frac{1}{3} \delta_{ij} \delta_{kl}, \quad (95)$$

For $q = 2$,

$$Q_{ijkl}^{(0;2)} = (a \delta_{ik} \delta_{jl} + b \delta_{il} \delta_{jk}), \quad (96)$$

According to orthogonality condition (the inner product of both equations (95) and (96) are taken)

$$Q_{ijkl}^{(0;1)} Q_{ijkl}^{(0;2)} = 0, \quad Q_{ijkl}^{(0;2)} Q_{ijkl}^{(0;2)} = 1 \quad (97)$$

so for $q = 2$, $a = \frac{1}{2\sqrt{3}}$, $b = -\frac{1}{2\sqrt{3}}$. The second idempotent is

$$Q_{ijkl}^{(0;2)} = \frac{1}{2\sqrt{3}} \delta_{ik} \delta_{jl} - \frac{1}{2\sqrt{3}} \delta_{il} \delta_{jk} \quad (98)$$

By using the same procedure, the last idempotent is found as

$$Q_{ijkl}^{(0;3)} = \frac{1}{6\sqrt{5}} (3\delta_{ik} \delta_{jl} + 3\delta_{il} \delta_{jk} - 2\delta_{ij} \delta_{kl}) \quad (99)$$

As an example to how we obtain the irreducible parts from idempotents is demonstrated, $C_{ijkl}^{(0;1)}$ is given

$C_{ijkl}^{(0;1)} = Q_{ijkl}^{(0;1)} Q_{ijkl}' C_{ijkl}$, this gives

$$C_{ijkl}^{(0;1)} = \frac{1}{9} \delta_{ij} \delta_{kl} C_{ppqq}.$$

(100)

This is the first decomposed part of scalar part. Second decomposed part $C_{ijkl}^{(0;2)}$ is found as

$C_{ijkl}^{(0;2)} = Q_{ijkl}^{(0;2)} Q_{ijkl}' C_{ijkl}$, which gives

$$C_{ijkl}^{(0;2)} = \frac{1}{90} (3\delta_{ik} \delta_{jl} + 3\delta_{il} \delta_{jk} - 2\delta_{ij} \delta_{kl}) (3C_{ppqq} - C_{ppqq}).$$

(101)

Using same procedure, the other parts can be found.

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