

**DATA COMPRESSION WITH $\Sigma\Pi$ -ALGORITHM FOR THE APPROXIMATION
BASED ON B-SPLINES**

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Abstract

This paper contains short description of $\Sigma\Pi$ -algorithm for the approximation of the function with two independent variables by the sum of products of one-dimensional functions. It is easy to describe the construction of the best $\Sigma\Pi$ -approximation with the given accuracy level. The structure of the construction of the best $\Sigma\Pi$ -approximation usually repeats this structure with some differences concerning with the algorithm for the eigenvalue problem and some additional possibilities like visual control. Some realizations of this algorithm based on B-splines are presented here. Few examples concerning with compression in the solving of approximation problems are described and discussed.

Key words: data compression, $\Sigma\Pi$ - algorithm, B-splines, approximation, the given accuracy level, eigenvalue problem.

1. INTRODUCTION

In last few years we can observe growing interest to digital methods of processing of signals and images. It is caused by that the digital methods of processing in comparison with analog are exacter, reliable, flexible and simple in realization. The interest to algorithms of processing of the images is initiated due to development of characteristics of computer facilities on the one hand, and occurrence of tasks connected to job on processing of the images on the other hand. For example, the widely widespread communication technologies recently are based on transfer of digital signals and images on the large distances in the distributed systems.

Data compression is one of the most significant components of the modern information processing systems. It is related to the development of digital measuring equipment, possibilities and comfort of using computers in processing, transferring and keeping incoming data. Meanwhile, size of kept and transferred information is usually large-scale. With that comes the problem of effective data compression, either lossless or with allowable level of loss.

There are lots of effective data compression methods at the moment. This algorithms encode data without any information losses. They are based on replacing repeated sequences with short codes (coding with Huffmann or Lempel-Ziv algorithms). This methods, though well-suited for text and monochrome image coding, are absolutely ineffective in real data (such as tables and experimental evidences), gray-scale and chromatic image compression.

This article suggests using $\Sigma\Pi$ - approximation algorithm, very effective bidimensional data compression method. This algorithm is based on approaching function with two variables by the sums of products of one-dimensional functions.

This method was suggested for the first time in Schmidt's paper in 1907. Schmidt has shown (E.Schmidt, 1907) that every square-integrated function of two variables $f(x,y)$ can be approximated in

average quadratic with desired accuracy level using finitely many terms, if each term represents product of function from x on function from y , also, the minimum of norm of an error of such expansion can be reached on Schmidt's fundamental functions (x, y) which correspond to singular values of the integrated operator with a kernel $f(x, y)$.

Optimization of these approximations in the average quadratic for functions of continuous or discrete argument was considered by various authors (E.Schmidt, (1907), M.R.Shura-Bura, (1957), V.V.Pospelov, (1978), R.D.Bagljaj and K.K.Smirnov, (1975)). The most general statement and computing algorithm of the solution were given by V.A.Vasilenko (1990).

The offered method of approximation can be used for compression of any two-dimensional information, for example, for compression of experimental data, for packing of tables, for video information compression. Thus the data is restored with the given accuracy level.

2. APPROACHES, TOOLS, DATA, DOMAINS AND NESTING

2.1. Theoretical background

In the modern formulation the problem of the best $\Sigma\Pi$ -approximation means the following. Let $X(\Omega_x)$ and $Y(\Omega_y)$ be two Hilbert functional spaces over multidimensional domains $\Omega_x \subset R^{n_x}$ и $\Omega_y \subset R^{n_y}$, $n_x \geq 1$, $n_y \geq 1$, $\Omega = \Omega_x \times \Omega_y$ and $Z(\Omega) = X(\Omega_x) \otimes Y(\Omega_y)$ be the tensor product of the spaces (W.A.Light & E.W.Cheney, 1985). For the function $f(x, y) \in Z(\Omega)$ we need to find the best $\Sigma\Pi$ -approximation in the form

$$\sum_{k=1}^s \Phi^{(k)}(x) \cdot \Psi^{(k)}(y), \quad (1)$$

where $\Phi^{(k)} \in X(\Omega_x)$, $\Psi^{(k)} \in Y(\Omega_y)$, or in the particular case (V.Vasilenko, 1990), they belong to the finite dimensional subspaces $X_n(\Omega_x)$ and $Y_m(\Omega_y)$ of the spaces $X(\Omega_x)$ and $Y(\Omega_y)$ respectively. For the unknown function $\Phi^{(k)}(x)$, $\Psi^{(k)}(y)$ we have the expansions

$$\Phi^{(k)}(x) = \sum_{i=1}^n \alpha_i^{(k)} \varphi_i(x), \quad n = \dim X_n, \quad (2)$$

$$\Psi^{(k)}(y) = \sum_{j=1}^m \beta_j^{(k)} \psi_j(y), \quad m = \dim Y_m,$$

where $\varphi_1, \varphi_2, \dots, \varphi_n$ form the basis in X_n , and $\psi_1, \psi_2, \dots, \psi_m$ form the basis in Y_m . Thus, to find the best $\Sigma\Pi$ -approximation we need to determine the coefficients $\alpha_i^{(k)}$, $\beta_j^{(k)}$, $i = \overline{1, n}$, $j = \overline{1, m}$, $k = \overline{1, s}$ from the minimization of the functional

$$E_{n,m}^{(s)}(\overline{\alpha}, \overline{\beta}) = \left\| f(x, y) - \sum_{k=1}^s \left[\sum_{i=1}^n \alpha_i^{(k)} \varphi_i(x) \sum_{j=1}^m \beta_j^{(k)} \psi_j(y) \right] \right\|_{Z(\Omega)}^2, \quad (3)$$

with respect to these coefficients. Without loss of generality we assume that

$$\|\Phi^{(k)}\|_X = \|\Psi^{(k)}\|_Y, \quad (\Phi^{(k)}, \Phi^{(l)})_X = (\Psi^{(k)}, \Psi^{(l)})_Y = 0, \quad k \neq l.$$

Let us introduce two Gram matrices

$$A = \{(\varphi_i, \varphi_j)_X\}_{i,j=1}^n, \quad B = \{(\psi_i, \psi_j)_Y\}_{i,j=1}^m,$$

and the rectangular $n \times m$ - matrix -F given by formula

$$F = \{f_{ij}\}_{i,j=1}^{n,m}, \quad f_{ij} = (f(x, y), \varphi_i(x)\psi_j(y))_Z,$$

and consider the generalized eigenvalue problem with the block matrices

$$\begin{bmatrix} 0 & F \\ F^* & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix}. \quad (4)$$

It is easy to see that the problem (4) has $s < \min\{n, m\}$ positive eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0,$$

and corresponding normalized linear independent eigenvectors

$$(Au^{(k)}, u^{(k)}) + (Bv^{(k)}, v^{(k)}) = 1, \quad k = \overline{1, s}$$

are connected with the optimal coefficients $\bar{\alpha}^{(k)}, \bar{\beta}^{(k)}$ by formula

$$\bar{\alpha}^{(k)} = \sqrt{2\lambda_k} u^{(k)}, \quad \bar{\beta}^{(k)} = \sqrt{2\lambda_k} v^{(k)}. \quad (5)$$

Taking into account the "a posteriori" error estimate

$$E_{n,m}^{(s)} = \|f\|_Z^2 - \sum_{k=1}^s \lambda_k^2. \quad (6)$$

one can see that the optimal strategy is the search of eigenvalues in order of decay to provide the best minimization of $E_{n,m}^{(s)}(\bar{\alpha}, \bar{\beta})$.

Using the well-known trick based on the Cholesky decomposition $A = LL^*, B = MM^*$ of the Gram matrices, it is possible to reduce the generalized eigenvalue problem (4) to usual eigenvalue problem

$$(L^{-1}FM^{*-1}) * (L^{-1}FM^{*-1})\omega = \lambda^2\omega \quad (7)$$

of the order n or m .

It is easy to describe the main steps in the construction of the best $\Sigma\Pi$ -approximation with the given accuracy level $\varepsilon > 0$ (O.E.Baklanova&V.A.Vasilenko,1993):

Step 1. Choice of the basic Hilbert spaces X and Y

Step 2. Definition of the Hilbert tensor product $X \otimes Y$ corresponding to the suitable cross-norm

Step 3. Choice of the finite dimensional subspaces $X_n \subset X$, $Y_m \subset Y$ and their basis

Step 4. Assembling of the Gram matrices A and B

Step 5. Assembling of the rectangular $n \times m$ matrix F

Step 6. Consequent determination of the eigenvalues and eigenvectors to provide the given accuracy level in $\Sigma\Pi$ -approximation

The structure of the software for the construction of the best $\Sigma\Pi$ - approximation usually repeats this structure with some differences concerning with the concrete algorithm for the eigenvalue problem and some additional possibilities like visual control.

3. APPLICATION VALIDATION

3.1. $\Sigma\Pi$ -approximation and polynomial splines

Let $\Omega_x = [a, b]$, $\Omega_y = [c, d]$, and $X(\Omega_x) = H^s(a, b)$, $Y(\Omega_y) = H^r(c, d)$ the Hilbert spaces with the scalar products

$$\begin{aligned} (u, v)_{H^s(a,b)} &= \int_a^b [uv + u^{(s)}v^{(s)}]dx \\ (u, v)_{H^r(c,d)} &= \int_c^d [uv + u^{(r)}v^{(r)}]dy \end{aligned} \quad (8)$$

The tensor product $H^{s,r}(\Omega)$, $\Omega = [a, b] \times [c, d]$ of these two spaces with respect to the suitable cross-norm consists of the function of 2 variables with the norm

$$\|u(x, y)\|_{H^{s,r}(\Omega)} = \left[\int_a^b \int_c^d \left[u^2 + \left(\frac{\partial^s u}{\partial x^s} \right)^2 + \left(\frac{\partial^r u}{\partial y^r} \right)^2 + \left(\frac{\partial^{s+r} u}{\partial x^s \partial y^r} \right)^2 \right] d\Omega \right]^{1/2}$$

and corresponding scalar product.

Let us divide the intervals $[a, b]$ and $[c, d]$ by some mesh points,

$$a = x_1 < x_2 < \dots < x_n = b, c = y_1 < y_2 < \dots < y_m = d$$

and connect with these meshes the finite dimensional subspaces $X_n = S_n \subset H^s(a, b), Y_m = S_m \subset H^r(c, d)$ of the polynomial splines of the defect one with the suitable smoothness. It is well-known fact that the bases of the local functions (B-splines) do exist in three spline spaces (C.De.Boor,1978). If the function $f(x,y)$ belongs to the space $H_{s,r}(\Omega)$, then the prescribed $\Sigma\Pi$ - procedure can be applied to approximate $f(x,y)$ by the sum of products of one-dimensional splines.

3.2. Using the discrete splines for data compression

If the initial two-dimensional function is given in the discrete form $f(i,j)$, like the image after scanning, then it is natural to apply the discrete splines instead continuous polynomial splines in $\Sigma\Pi$ - algorithm of data compression. Let

$$\Omega_x = \{1, 2, \dots, n\}, \quad \Omega_y = \{1, 2, \dots, m\}.$$

We denote by $H_s(\Omega_x)$ and $H_r(\Omega_y)$ the Hilbert spaces of the mesh functions with the norms

$$\|u\|_{H^s(\Omega_x)} = \left[\frac{1}{n} \sum u^2(i) + \frac{1}{n-s} \sum [(\Delta_s u)(i)]^2 \right]^{1/2}$$

$$\|v\|_{H^r(\Omega_y)} = \left[\frac{1}{m} \sum v^2(i) + \frac{1}{m-r} \sum [(\Delta_r v)(i)]^2 \right]^{1/2}$$

where Δ_s, Δ_r mean the divided differences of the order n or m . The tensor product $H_{s,r}(\Omega_x \times \Omega_y)$ can be defined as the Hilbert space of two-dimensional mesh function with the cross-norm

$$\|u\|_{H^s(\Omega_x)} = \left[\frac{1}{nm} \sum u^2(i, j) + \frac{1}{(n-s)m} \sum (\Delta_s u)^2(i, j) \right. \\ \left. + \frac{1}{n(m-r)} \sum (\Delta_r u)^2(i, j) \right. \\ \left. + \frac{1}{(n-s)(m-r)} \sum (\Delta_s \Delta_r u)^2(i, j) \right]^{1/2}$$

where Δ_s and Δ_r mean the finite difference differentiation with respect to 1-st and 2-nd variable.

Let $\omega_x \geq 1, \omega_y \geq 1$ be some integers, and $n = k_x \cdot \omega_x, m = k_y \cdot \omega_y$, where k_x, k_y are also integer. It is easy to connect with the "uniform mesh" $1, \omega, 2\omega, \dots, k\omega$ the space of discrete splines with the basis of discrete B-splines; these B-splines can be obtained by the few discrete convolutions of the mesh piecewise constant function

$$\chi(i) = \begin{cases} 1, & -\left[\frac{\omega}{2}\right] \leq i \leq \left[\frac{\omega}{2}\right] \\ 0, & \text{otherwise.} \end{cases}$$

3.3. Compression of colour images

In the red-green-blue (RGB) representation the colour image is the vector function of these components. At first we extract from the colour image the half-tone component by the following law:

every colour pixel (r_i, g_i, b_i) corresponds to the scalar $a_i = \min\{r_i, g_i, b_i\}$. The rest part of colour image consists of the pixels $(r_i - a_i, g_i - a_i, b_i - a_i)$. Formally we divide the initial colour image into four half-tone components $I_1 = \{a_i\}$, $I_2 = \{r_i - a_i\}$, $I_3 = \{g_i - a_i\}$, $I_4 = \{b_i - a_i\}$. Each of these components is the scalar mesh function of two integer variables, and for each of this functions we apply $\Sigma\Pi$ -algorithm based on discrete one-dimensional splines with $\omega = 1$. In this case the "degree" of the splines does not matter, and the norm which we choose is $H_{0,0}$ -norm.

There is no way to demonstrate the compressed and the initial colour image in this edition because colour photographs are impossible here. By these means we give only an expert evaluations for some colour 320 x 200- images. To provide the exact restoration of this image we need 200 terms in $\Sigma\Pi$ -approximation. But the visual control shows that the number of terms is extremely less, especially for the components I_2, I_3, I_4 . We show the expert evaluation of the colour image quality in the following table 1.

Table 1. The expert evaluation of the colour image quality

Number of eigenvalues				Expert evaluation
I1	I2	I3	I4	
50	1	1	1	fair
50	3	2	3	good
50	7	6	10	excellent

4. SOME EXAMPLES FROM THE NUMERICAL EXPERIMENTS

4.1. The examples of $\Sigma\Pi$ -approximation and polynomial splines

To estimate the compression coefficient we consider bicubic spline $f(x,y)$ connected with the uniform 6x6 rectangular grid in the unit square which interpolates the function

$$f(x, y) = \sin xy \cdot \exp(x^2 + y^2)$$

with the accuracy 10^{-3} in $H_{2,2}$ -norm (see Fig. 1). The exact representation of this spline with two-dimensional B-splines requires $8 \times 8 = 64$ coefficients. If we construct $\Sigma\Pi$ -approximation based on the one-dimensional cubic splines with the same accuracy level 10^{-3} in $H_{2,2}$ -norm, then only 2 eigenvalues are necessary to provide this error, and a number of coefficients in $\Sigma\Pi$ -approximation is $2 \times 8 \times 2 = 32$; thus the compression coefficient is $64/32 = 2$.

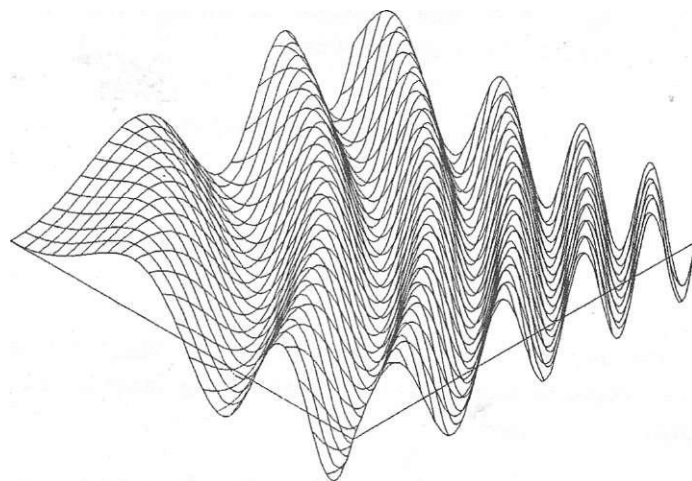


Fig. 1 The bicubic spline with coefficients

$$b_{ij} = \sin^3 \left(10 \left(\frac{i-1}{31} \right)^2 + 10 \left(\frac{j-1}{31} \right)^2 \right)$$

4.2. The examples of using the discrete splines for data compression

We describe now an example of data compression with the discrete splines. Let the function

$$f(x, y) = \sin(4\pi(x^2 + y^2 + xy))$$

be given at the unit square $[0, 1] \times [0, 1]$ (see Fig. 2).

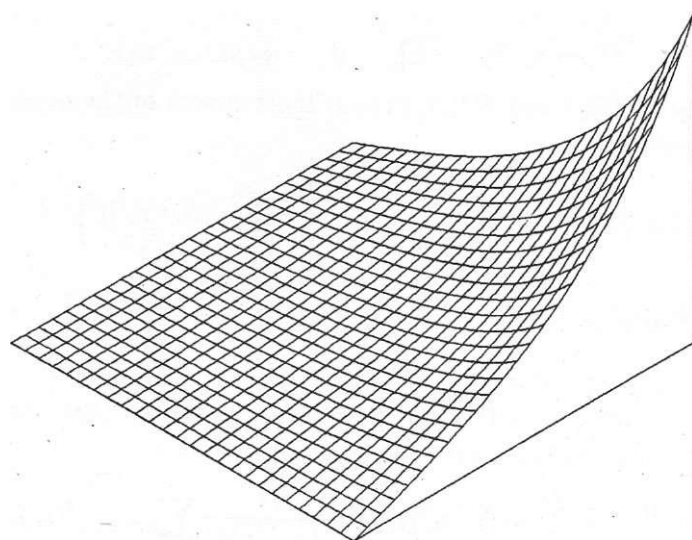


Fig. 2. $\Sigma\Pi$ -approximation for example of using the discrete splines for data compression

We consider the corresponding mesh function at 130×130 uniform grid and approximate it with the accuracy level 0.01 in the mesh $H_{2,2}$ -norm by the $\Sigma\Pi$ -function with one dimensional discrete splines, corresponding to $\omega = 3$. We need only nine terms to provide this accuracy, and compression coefficient in this case is about 41.4.

5. CONCLUSIONS

For experimental data compression were developed specific computing circuits of compression with needed accuracy level based on $\Sigma\Pi$ - approximation algorithm using continuous and discrete splines. $\Sigma\Pi$ - approximation algorithm was formulated on the basis of general methods of approaching which allows to considerably extending the class of functional dependencies and estimation types for receipt of desired accuracy.

So, in this paper was examined the task of continuous and discrete spline dependencies from two variables compression. As generally know, polynomial spline from two variables storing also needs storing its expansion factors for every cell of the grid. If this grid contains too many nodes, information size can be pretty large. And also, in practice estimating the value of the spline is often needed for the point with low accuracy. In this situation $\Sigma\Pi$ - approximation algorithm allows to replace the spline of two variables by a few one-dimensional splines, and thus considerably reduce the amount of stored factors. Moreover, $\Sigma\Pi$ - approximation algorithm is a bit easier for this task.

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