

STRENGTH DESIGN WITH ORIENTATION ANGLE DESIGN VARIABLES

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Abstract

The design of a composite laminate is a combinatorial problem with layer thickness and orientation angles design variables. The majority of the commercially available optimization software is for continuous-valued design variables with orientations taking any value between 0° and 90°. The optimization problem with strength constraints were formulated to minimize the laminate weight. In weight minimization problem, orientation variables influence only the strength constraints.

Key words: laminated 2D structures, strength design, sensitivity analysis, orientation angles design variables

1. INTRODUCTION

The unidirectional fibrous composite materials are the directional dependence of the strength on a macroscopic scale. Composite layers are much stronger in the fiber direction than in the direction perpendicular to the fibers. For loads that are primarily parallel to the fibers, either in the tension or compression, the material strength is generally governed by the failure of the fibers. For loads transverse to the fibers, failure is controlled by the failure of the much weaker matrix material. In order to find a laminate stacking sequence that is best situated to a specified loading, the orientation angles of the layers can be used as a design variables.

2. CLASSICAL LAMINATION THEORY

Laminate structures are typical lightweight elements with expanding application in civil and mechanical engineering. For the modelling and analysis of laminates we used the classical laminate theory (CLT) [1]. The CLT is an extension of Kirchhoff's classical plate theory for homogeneous isotropic plates to laminated composite plates with a high width-to-thickness ratio.

The assumptions for macro-mechanical modelling of laminate are [2]:

1. All layers are in a state of plane stress, i. e.:

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0. \quad (1)$$

2. Normal distances from the middle surface remain constant.
3. The transverse shear strains γ_{xz}, γ_{yz} are negligible.

In-plane strains can be noted as:

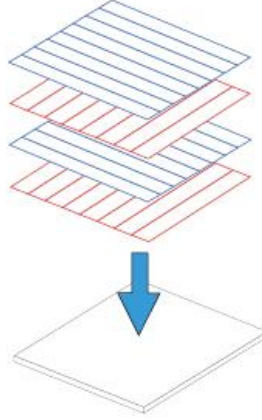
$$\boldsymbol{\varepsilon}(x, y, z) = \bar{\boldsymbol{\varepsilon}}(x, y) + z \boldsymbol{\kappa}. \quad (2)$$

where $\bar{\epsilon}$ is the vector of mid-plane strains and κ is the vector of curvatures, that are constant through the thickness.

The stress resultant force vectors are:

$$\begin{aligned} N &= \int_{-h/2}^{+h/2} E(z) dz \bar{\epsilon} + \int_{-h/2}^{+h/2} E(z) z dz \kappa, \\ M &= \int_{-h/2}^{+h/2} E(z) z dz \bar{\epsilon} + \int_{-h/2}^{+h/2} E(z) z^2 dz \kappa. \end{aligned} \quad (3)$$

Figure 1. Laminate stacking sequence



The equilibrium equations are yield three force and two moment

formulated for a plate element and equations [4]:

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + p_1 &= 0, & \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_2 &= 0, \\ \frac{\partial V_{xz}}{\partial x} + \frac{\partial V_{yz}}{\partial y} + p_3 &= 0, \end{aligned} \quad (4)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = V_{xz}, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = V_{yz}.$$

For the modeling of laminate it is assumed that each individual layer has property of a linear-elastic material, all layers are bonded together with a perfect bond and each lamina of composite material behaves macroscopically as a homogeneous orthotropic material.

The constitutive equations can be written in the condensed hypermatrix form:

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} \bar{\epsilon} \\ \kappa \end{pmatrix} \quad (5)$$

with:

$$\{A, B, D\} = \int_{-h/2}^{+h/2} E(z) \{1, z, z^2\} dz. \quad (6)$$

The elasticity matrix we can write as:

$$E = \sum_{n=1}^N \frac{n h}{h} E, \quad {}^n E = \bar{T}^T({}^n \theta) E_L \bar{T}({}^n \theta), \quad \bar{T}(\theta) = (T^T(\theta))^{-1}, \quad (7)$$

where T is the transformation matrix.

The stress-strain relation ship in the local coordinate system can be noted as:

$$\sigma_L = {}^{(n)}E_L \varepsilon_L \Leftrightarrow \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} (E_{11})_L & (E_{12})_L & 0 \\ & (E_{22})_L & 0 \\ sym. & & (E_{44})_L \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix}, \quad (8)$$

where:

$$\begin{aligned} (E_{11})_L &= \frac{E_1}{1 - \nu_{12} \cdot \nu_{21}}, & (E_{22})_L &= \frac{E_2}{1 - \nu_{12} \cdot \nu_{21}}, \\ (E_{12})_L &= \nu_{12} (E_{22})_L, & (E_{44})_L &= G_{12}. \end{aligned} \quad (9)$$

3. COMPUTATIONAL MICROMECHANICS

However, most composites have random arrangement of the fibers. A random microstructure results in transversely isotropic properties at the meso-scale. A simple alternative is to assume that the random microstructure is well approximated by the hexagonal microstructure. The elastic properties of the homogenized material can be computed by [3]:

$$\begin{aligned} E_1 &= C_{11} - 2C_{12}^2 / (C_{22} + C_{23}), \\ \nu_{12} &= C_{12} / (C_{22} + C_{23}), \\ E_2 &= (C_{11}(C_{22} + C_{23}) - 2C_{12}^2)(C_{22} - C_{23}) / (C_{11}C_{22} - C_{12}^2), \\ G_{12} &= C_{66}. \end{aligned} \quad (10)$$

In order to evaluate the elastic matrix $\mathbf{C}$ of the composite, the representative volume element (RVE) is subjected to an average strain. The volume average of the strain in the RVE equals the applied strain:

$$\bar{\varepsilon}_{ij} = \frac{1}{V} \int_V \varepsilon_{ij} dV. \quad (11)$$

The components of the tensor $\mathbf{C}$ are determined solving three elastic models of RVE with parameters (a_1, a_2, a_3) subjected to the boundary conditions. By using a unit value of applied strain, and the problem is solved, it is possible to compute the stress field, whose average gives the required components of the elastic matrix as:

$$C_{ij} = \bar{\sigma}_i = \frac{1}{V} \int_V \sigma_i dV. \quad (12)$$

The coefficients in $\mathbf{C}$ are found by setting a different problem for each column of $\mathbf{C}$.

In order to determine the components C_{i1} with $i = 1, 2, 3$, the following strain is applied to stretch the RVE in the fiber direction x_1 .

$$\varepsilon_1^0 = 1,$$

$$\varepsilon_2^0 = \varepsilon_3^0 = \gamma_4^0 = \gamma_5^0 = \gamma_6^0 = 0. \quad (13)$$

The following boundary conditions on displacements can be used:

$$\begin{aligned} u_1(a_1, x_2, x_3) &= a_1, \\ u_1(0, x_2, x_3) &= 0, \\ u_2(x_1, a_2, x_3) &= 0, \\ u_2(x_1, 0, x_3) &= 0, \\ u_3(x_1, x_2, a_3) &= 0, \\ u_3(x_1, x_2, 0) &= 0. \end{aligned} \quad (14)$$

The coefficients C_{i1} are found by using:

$$C_{i1} = \bar{\sigma}_i. \quad (15)$$

In order to determine the components C_{i2} with $i = 1, 2, 3$, the following strain is applied to stretch the RVE in the direction x_2 .

$$\begin{aligned} \varepsilon_2^0 &= 1, \\ \varepsilon_1^0 = \varepsilon_3^0 = \gamma_4^0 = \gamma_5^0 = \gamma_6^0 &= 0. \end{aligned} \quad (16)$$

The following boundary conditions on displacements can be used:

$$\begin{aligned} u_1(a_1, x_2, x_3) &= 0, \\ u_1(0, x_2, x_3) &= 0, \\ u_2(x_1, a_2, x_3) &= a_2, \\ u_2(x_1, 0, x_3) &= 0, \\ u_3(x_1, x_2, a_3) &= 0, \\ u_3(x_1, x_2, 0) &= 0. \end{aligned} \quad (17)$$

The coefficients C_{i2} are found by using the next equation:

$$C_{i2} = \bar{\sigma}_i. \quad (18)$$

In order to determine the components C_{i3} with $I = 1, 2, 3$, the following strain is applied to stretch the RVE in the direction x_3 .

$$\begin{aligned} \varepsilon_3^0 &= 1, \\ \varepsilon_1^0 = \varepsilon_2^0 = \gamma_4^0 = \gamma_5^0 = \gamma_6^0 &= 0. \end{aligned} \quad (19)$$

The following boundary conditions on displacements can be used:

$$\begin{aligned}
 u_1(a_1, x_2, x_3) &= 0, \\
 u_1(0, x_2, x_3) &= 0, \\
 u_2(x_1, a_2, x_3) &= 0, \\
 u_2(x_1, 0, x_3) &= 0, \\
 u_3(x_1, x_2, a_3) &= a_3, \\
 u_3(x_1, x_2, 0) &= 0.
 \end{aligned} \tag{20}$$

The coefficients C_{i3} are found by using:

$$C_{i3} = \bar{\sigma}_i. \tag{21}$$

The coefficient C_{66} can be determined as follows:

$$C_{66} = \frac{1}{2}(C_{22} - C_{23}). \tag{22}$$

The fiber volume fraction is expressed:

$$v_f = \frac{\pi}{\sqrt{3}} \left(\frac{d^2}{2a^2} \right). \tag{23}$$

The minimum thickness of each layer is:

$$t = d \left[\left(\frac{\pi}{2\sqrt{3}v_f} \right)^{0.5} - 1 \right]. \tag{24}$$

4. FAILURE CRITERIA FOR 2D UNIDIRECTIONAL LAMINATE

We can distinguish the failure between fiber failure (FF) and inter fiber failure (IFF). In the case of plane stress, the IFF criteria discriminates three different modes. The IFF mode A is when perpendicular transversal cracks appear in the lamina under transverse tensile stress with or without in-plane shear stress. The IFF mode B denotes perpendicular transversal cracks, but in this case they appear under in-plane shear stress with small transverse compression stress. The IFF mode C indicates the onset of oblique cracks when the material is under significant transversal compression [9].

Strength of a composite layer in any other direction is evaluated based on various failure criteria. The basic premise in predicting the failure of fiber-reinforced layers using maximum stress and maximum strain criteria is the same as for isotropic material. Failure is predicted when the maximum stress along the fiber or transverse to the fiber directions exceed the strength of the tension or compression .

The failure envelope is defined by:

$$\sigma_1 < X_t, \quad \sigma_2 < Y_t \quad \text{for} \quad \sigma_1, \sigma_2 > 0,$$

$$\sigma_1 > -X_c, \quad \sigma_2 > -Y_c \quad \text{for} \quad \sigma_1, \sigma_2 < 0, \quad (25)$$

$$|\tau_{12}| < S,$$

where:

X and Y represent the ultimate strengths along and transverse to the fiber directions, respectively. S is the ultimate in-plane shear strength of a specimen under pure shear loading. The maximum strain criterion similarly states that failure occurs when one of the following inequalities is violated:

$$\begin{aligned} \varepsilon_1 < \frac{X_t}{E_1}, \quad \varepsilon_2 < \frac{Y_t}{E_2} \quad \text{pre} \quad \varepsilon_1, \varepsilon_2 > 0, \\ \varepsilon_1 > -\frac{X_c}{E_1}, \quad \varepsilon_2 > -\frac{Y_c}{E_2} \quad \text{pre} \quad \varepsilon_1, \varepsilon_2 < 0, \\ |\gamma_{12}| < \frac{S}{G_{12}}. \end{aligned} \quad (26)$$

Tsai-Wu criterion is the general form of the failure criterion for orthotropic materials under plane stress. The assumption is expressed as:

$$F_{01}\sigma_1 + F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{02}\sigma_2 + F_{22}\sigma_2^2 + F_{44}\tau_{12}^2 < 1, \quad (27)$$

where:

$$\begin{aligned} F_{01} &= \frac{1}{X_t} - \frac{1}{X_c}, \quad F_{11} = \frac{1}{X_t X_c}, \quad F_{02} = \frac{1}{Y_t} - \frac{1}{Y_c}, \\ F_{22} &= \frac{1}{Y_t Y_c}, \quad F_{12} = -\frac{1}{2\sqrt{X_t X_c Y_t Y_c}}, \quad F_{44} = \frac{1}{S^2}. \end{aligned} \quad (28)$$

The failure criterion for orthotropic material under strain assumption is expressed as:

$$G_{01}\varepsilon_1 + G_{11}\varepsilon_1^2 + G_{12}\varepsilon_1\varepsilon_2 + G_{02}\varepsilon_2 + G_{22}\varepsilon_2^2 + G_{44}\gamma_{12}^2 < 1, \quad (29)$$

where:

$$G_{01} = F_{01}E_{11} + F_{02}E_{12} \quad G_{02} = F_{02}E_{22} + F_{01}E_{12}$$

$$G_{11} = F_{11}E_{11}^2 + F_{22}E_{12}^2 + F_{12}E_{11}E_{12} \quad G_{22} = F_{22}E_{22}^2 + F_{11}E_{12}^2 + F_{12}E_{22}E_{12}$$

$$G_{12} = 2E_{12}(F_{11}E_{11} + F_{22}E_{22}) + 2F_{12}(E_{12}^2 + E_{11}E_{22}) \quad G_{44} = F_{44}E_{44}^2. \quad (30)$$

When $F_{12} = \frac{-1}{2X_t^2}$, the Tsai-Wu criterion is reduced to Tsai-Hill criterion, and when $F_{12} = \frac{-1}{2X_tX_c}$ the Tsai-Wu criterion is reduced to Hoffman criterion [5].

These failure criteria are used to calculate a failure index (F.I.) from the computed stresses and user-supplied material strengths. A failure index denotes the onset of failure, and a value less than 1 denotes no failure. The failure index according to this theory is computed using the following equation:

$$I_F = F_{01}\sigma_1 + F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{02}\sigma_2 + F_{22}\sigma_2^2 + F_{44}\tau_{12}^2 \quad (31)$$

5. SOLUTION, RESULTS AND DISCUSSION

Solve the sensitivity of a 6-layer $[-\theta/0/\theta]_s$ Graphite-Epoxy laminate due to changes in the ply orientation θ . The thickness of the laminate is $0.125 \cdot 10^{-3} \text{m}$. The material constatnts are given as:

$E_1 = 181 \text{GPa}$, $E_2 = 10.3 \text{GPa}$, $G_{12} = 7.17 \text{GPa}$, $\nu_{12} = 0.28$

$X_t = 1500 \text{MPa}$, $X_c = 1500 \text{MPa}$, $Y_t = 40 \text{MPa}$, $Y_c = 246 \text{MPa}$, $S = 68 \text{MPa}$.

The bending loading is applied to the 2D structure $q = 2 \text{kPa}$.

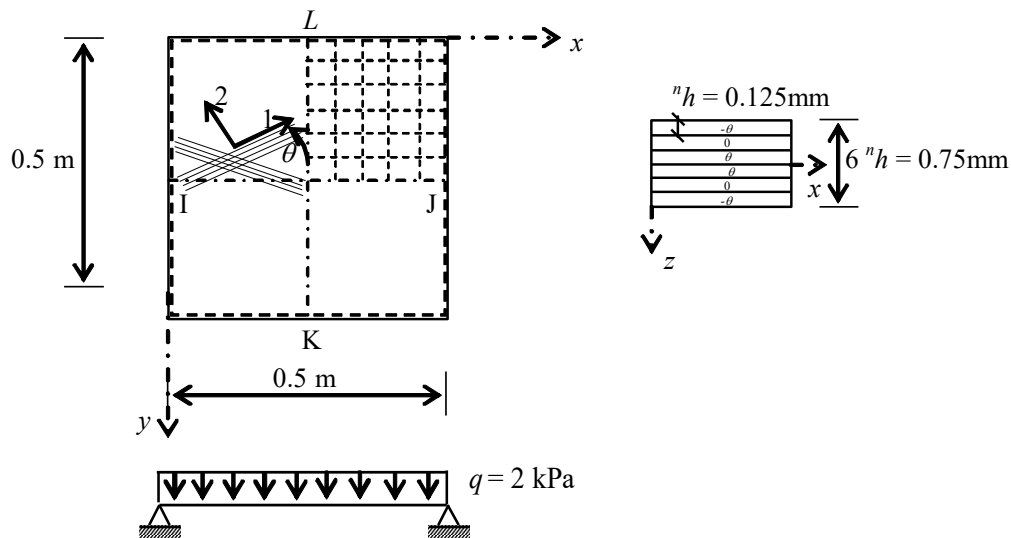


Figure 2. Geometry of the laminate plate

Solving:

Table 1. The failure index versus design variable

Design value	The failure index
0	0.8411570
10	0.8461421
20	0.9264359

30	0.9725791
40	0.9784101
50	0.9704180
60	0.9490395
70	0.8867959
80	0.9067153
90	0.9100196

In the orthotropic layers of the laminate there is a large difference of stiffness and strength between the two principal material direction in a layer [8]. In a laminate made of a mix of various fiber orientation angles and loaded along the x -coordinate axis, 0° layers, which are oriented along the x axis, owing to their large stiffness carry most of the applied loads. For a Graphite-Epoxy laminate $E_1/E_2 = 15.4$ with 50% 0° and 50% 90° layers, almost 94% of the loads will be carried by 0° layers. Under these circumstances, failure of a load-carrying layer may dictate the failure of the laminate. In Figure 3 there is shown, that failure is predicted when the orientation of angle is $\theta = 40^\circ$. Design variable is changed from (0°) to (90°) in ten optimization steps. The values of the failure index are from 0.84116 to 0.97841. The maximum failure index is 0.9784101. When it would use more sensitivity steps we would obtain exact value for the orientation angle design variable. The example was solved in the program COSMOS [6,7].

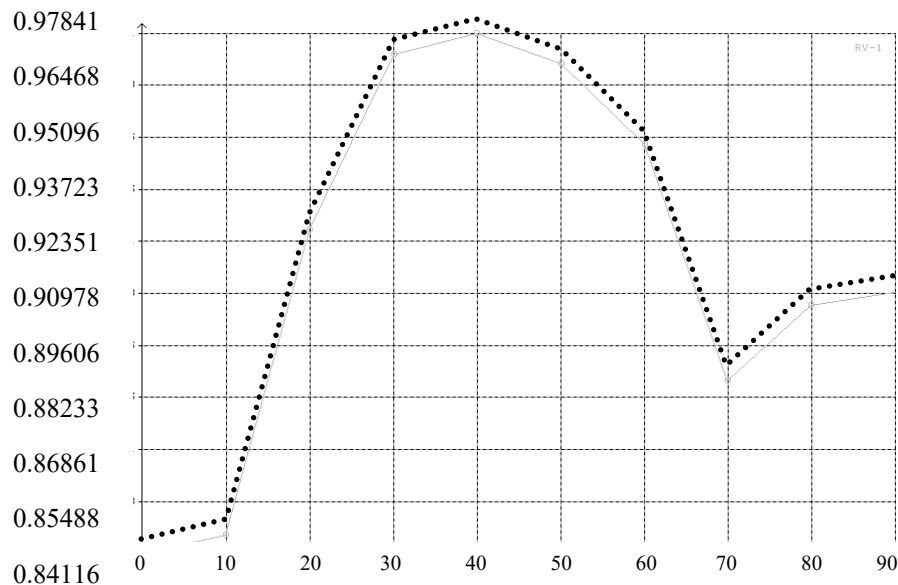


Figure 3. Failure index versus orientation of angle

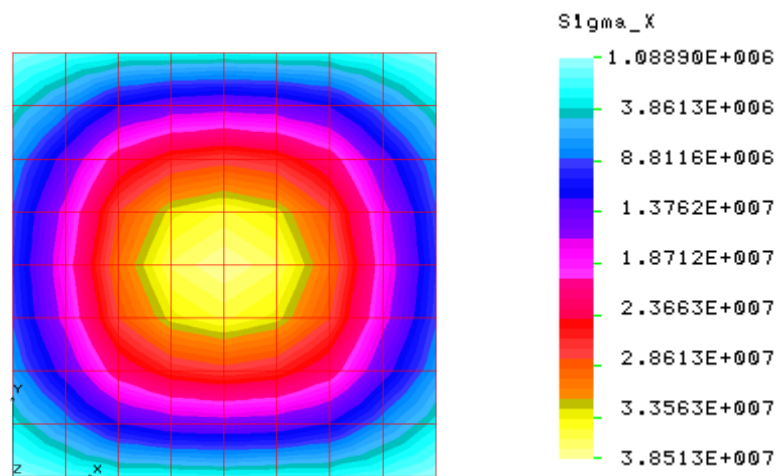


Figure 4. The stresses σ_x [Pa] for the laminate plate $[-\theta/0/\theta]_s$ with $\theta = 90^\circ$

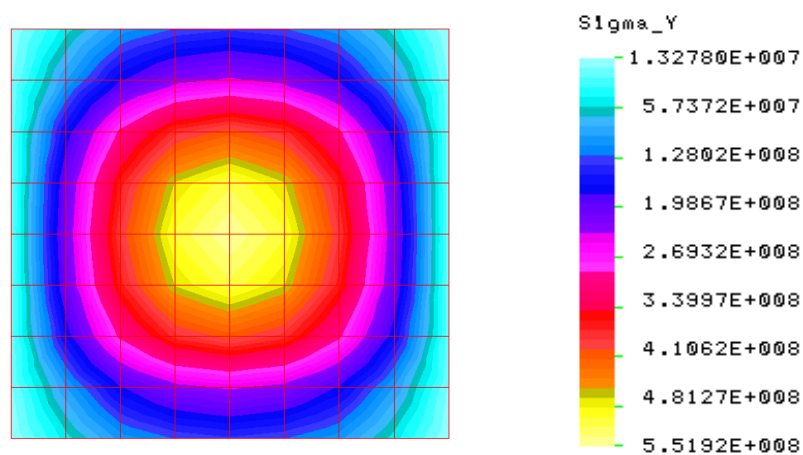


Figure 5. The stresses σ_y [Pa] for the laminate plate $[-\theta/0/\theta]_s$ with $\theta = 90^\circ$

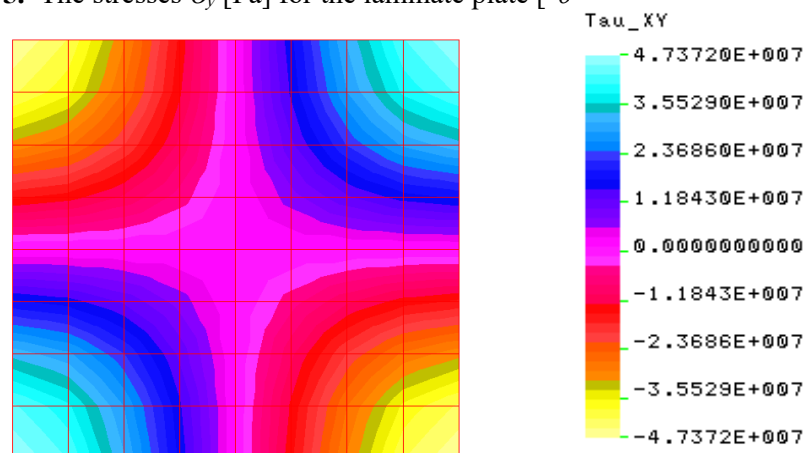


Figure 6. The stresses τ_{xy} [Pa] for the laminate plate $[-\theta/0/\theta]_s$ with $\theta = 90^\circ$

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