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A CLASS OF SPARSE MATRICES WITH WEIGHTS ENCODING LOCAL INFORMATION

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Abstract

We introduce a class of sparse matrices with weights $U_m(A_{p_1})$ of order m by m, where m is a composite natural number, p_1 is a divisor of m and A_{p_1} is a set of non zero real numbers, which we call weights. Each matrix $U_m(A_{p_1})$ has all entries 0 except for a small number of entries that belongs to the set A_{p_1}. The construction of $U_m(A_{p_1})$ is achieved by iteration and the invertible transform induced by $U_m(A_{p_1})$ is suitable for revealing information at different scales.

Key words: *sparse matrices with weights, linear transform, local information.*

1. INTRODUCTION

Nowadays, researchers deal with various data sets that appear as timeseries, sequences of observations taken sequentially in time. An intrinsic feature of a timeseries is that when we want to analyse, model and characterize it we require that some of the observations have different weights than others. For an example, in many cases, the most resent observations affect more the evolution of a certain time series, than the past observations (for more see [3]).

In order to deal with this problem we try to provide efficient multiscale analysis on finite data, so we seek for linear transforms whose corresponding matrices have the ability to detect specific characteristics from those data. In [1], we introduced a class of weighted sparse matrices for the purpose of prediction of almost periodic timeseries.

In this work we present the ability of the sparse matrices with weights to encode local information. We demonstrate the construction of these matrices. Each matrix results from an initial matrix with the process of dilation and contexture in multiple replicas in order to produce new matrices that maintain some of the basic properties of the initial matrix. In this way, we produce a multiscale analysis that provides us to encode local information. For example, a matrix of order *3 x 3* can be evolved into a matrix of order 3^n *x* 3^n, $n = 2, 3,..,$ therefore a timeseries of length 3^n, can be coded/compressed in a timeseries of length 3^k, where $k<n$, and because these matrices are sparse we achieve fast computations (see [2]).

In Section 2, we recall the construction of the weighted matrices by iteration, involving repetitive dilation operations and block-matrix operations. In Theorem 1 and 2 we prove two new basic properties of the matrices that are induced by their inductive construction and in Proposition 1 we recall the structure of the inverse matrices.

In section 3, we present the linear transform induced by the weighted matrices, we present the multiscale analysis that it is generated and we demonstrate its use in coding local information.

2. A CLASS OF SPARSE MATRICES WITH WEIGHTS

Notation: Let $\mathcal{M}_{n,m}$ be the set of all $n \times m$ matrices over the field of real numbers. If $n = m$, then $\mathcal{M}_{n,m}$ is abbreviated to $\mathcal{M}_n$. We shall use the symbolism $A = [A_{ij}]$ to denote a matrix A with elements A_{ij}. The notation $A_i = \{A_{i,j} : j = 1,...,m\}$ shall be used to denote the i row of a matrix $A \in \mathcal{M}_{n,m}$. The ***ceiling*** of a real number x shall be denoted by $\lceil x \rceil = inf\{n \in \mathbf{Z}: x \leq n\}$ ($\mathbf{Z}$ is the set of integers). If p,q are natural numbers we denote by $Mod(p,q)$ the remainder on division of p by q. We define the following operators:

Definition 1 Let D_p: $\mathcal{M}_{n,m} \to \mathcal{M}_{n,pm}$, such that:

$$D_p(M) = \left\{ M_{i,\left\lceil \frac{j}{p} \right\rceil},\ i = 1,...,n,\ \ j = 1,...,pm \right\}.$$

Examples: $D_2\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11}\, a_{11}\, a_{12}\, a_{12} \\ a_{21}\, a_{21}\, a_{22}\, a_{22} \end{pmatrix}$, $D_3\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11}\, a_{11}\, a_{11}\, a_{12}\, a_{12}\, a_{12} \\ a_{21}\, a_{21}\, a_{21}\, a_{22}\, a_{22}\, a_{22} \end{pmatrix}$.

Definition 2 Let $S(.,...,.)$: $\mathcal{M}_{n_1,m} \times ... \times \mathcal{M}_{n_k,m} \to \mathcal{M}_{n_1+...+n_k,m}$ be the following block matrix operation:

$$S(M_1,...,M_k) = \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix}.$$

Definition 3 Let p, w be positive integers such that w is a divisor of p and let $A_w = \{a_1,...,a_w\}$ be a set of non zero real numbers. We define the following block matrix operator: $Q_{p,A_w} : \mathcal{M}_{n,m} \to \mathcal{M}_{pn,pm}$:

$$Q_{p,A_w}(M) = \underbrace{a_1M \oplus ... \oplus a_1M}_{p/w-times} \oplus ... \oplus \underbrace{a_wM \oplus ... \oplus a_wM}_{p/w-times} = \begin{pmatrix} a_1M & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & a_wM \end{pmatrix},$$

where $\oplus$ is the **direct sum** of matrices and $\boldsymbol{O}$ is the zero matrix of order n x m.

Example:

$$Q_{3,\{a_1,a_2,a_3\}}\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)=\begin{pmatrix}a_1\begin{pmatrix}1\\-1\end{pmatrix} & \boldsymbol{O} & \boldsymbol{O}\\ \boldsymbol{O} & a_2\begin{pmatrix}1\\-1\end{pmatrix} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O} & a_3\begin{pmatrix}1\\-1\end{pmatrix}\end{pmatrix}=\begin{pmatrix}a_1 & 0 & 0\\ -a_1 & 0 & 0\\ 0 & a_2 & 0\\ 0 & -a_2 & 0\\ 0 & 0 & a_3\\ 0 & 0 & -a_3\end{pmatrix},$$ where $\boldsymbol{O}$ is the zero matrix of order *2 x 1*.

Definition 4 Let *p, q, w* be positive integers such that $q > 1$ and *w* is a divisor of *p*. Let $A_w = \{a_1,...,a_w\}$ be a set of non zero real numbers. We define the following matrix: $R(p,q,A_w) \in \mathcal{M}_{p(q-1),pq}$:

$$R(p,q,A_w)=\begin{cases} S\left(a_1\cdot \boldsymbol{e}_1^q,...,a_{q-1}\cdot \boldsymbol{e}_{q\text{-}1}^q\right), & p=1\\ S\left(Q_{p,A_w}\left(\boldsymbol{e}_1^q\right),...,Q_{p,A_w}\left(\boldsymbol{e}_{q\text{-}1}^q\right)\right), & p>1\end{cases},$$

where $\boldsymbol{e}_i^q$ is the *i* row of the identity matrix $\boldsymbol{I}_q$ and Q_{p,A_w} is given in definition 3 .

Example: Let $w = 2$ (i.e. $A_2 = \{a_1,a_2\}$), then:

$$R(2,3,A_2)=S\left(Q_{2,\{a_1,a_2\}}\left(\boldsymbol{e}_1^3\right),Q_{2,\{a_1,a_2\}}\left(\boldsymbol{e}_2^3\right)\right)=\begin{pmatrix}a_1\boldsymbol{e}_1^3 & \boldsymbol{O}\\ \boldsymbol{O} & a_2\boldsymbol{e}_1^3\\ a_1\boldsymbol{e}_2^3 & \boldsymbol{O}\\ \boldsymbol{O} & a_2\boldsymbol{e}_2^3\end{pmatrix}=\begin{pmatrix}a_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & a_2 & 0 & 0\\ 0 & a_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a_2 & 0\end{pmatrix},$$

where $\boldsymbol{e}_1^3=\{1,0,0\}$, $\boldsymbol{e}_2^3=\{0,1,0\}$ and $\boldsymbol{O}=\{0,0,0\}$.

From now on, we consider a composite positive integer $m = l\,p_1$, where $l \geq 1$. Moreover, we assume that *m* can be written as:

$$m=p_1p_2...p_N$$

$$J(0)=1,\ J(n)=\prod_{r=1}^{n}p_r,\quad n=1,...,N \tag{1}$$

$$A(i)=\prod_{r=i}^{N}p_r,\ i=1,...,N,\quad A(N+1)=1,$$

where $p_2 \geq p_3 \geq \ldots \geq p_N$ are prime factors of m/p_1. Notice that p_1 is not necessarily prime and that $J(n)$ and $A(i)$ are the partially products of the factors of m.

Definition 5 We consider the factorization (1) of a positive integer m and we define a sequence of block matrices $U_m\left(n, A_{p_1}\right) \in \mathcal{M}_{J(n)}$, where $n = 0,...,N$, by using the following iteration:

$$U_m\left(n, A_{p_1}\right) = \begin{cases} \begin{pmatrix} a_1 & \ldots & a_{p_1} \end{pmatrix}, & n = 0 \\ S\left(U_m\left(0, A_{p_1}\right), R\left(1, p_1, A_{p_1}\right)\right), & n = 1 \\ S\left(D_{p_n}\left(U_m\left(n-1, A_{p_1}\right)\right), R\left(J\left(n-1\right), p_n, A_{p_1}\right)\right), & n = 2,..., N \end{cases},$$

where $A_{p_1} = \left\{a_1, \ldots, a_{p_1}\right\}$ is as set of non zero real numbers. In case where $n = N$ we shall write: $U_m\left(N, A_{p_1}\right) = U_m\left(A_{p_1}\right)$.

Example: Let $m = 12$, take $p_1 = 3$, then $m = p_1 p_2 p_3$, where $p_2 = 2$, $p_3 = 2$, so:

$$U_{12}\left(0, A_3\right) = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}, \quad U_{12}\left(1, A_3\right) = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}, \quad U_{12}\left(2, A_3\right) = \begin{pmatrix} a_1 & a_1 & a_2 & a_2 & a_3 & a_3 \\ a_1 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_2 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 \end{pmatrix} \text{ and}$$

$$U_{12}\left(3, A_3\right) = \begin{pmatrix} a_1 & a_1 & a_1 & a_1 & a_2 & a_2 & a_2 & a_2 & a_3 & a_3 & a_3 & a_3 \\ a_1 & a_1 & a_1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & a_2 & a_2 & a_2 & 0 & 0 & 0 & 0 \\ a_1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & a_3 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 \end{pmatrix}.$$

Theorem 1 *The matrix* $U_m\left(A_{p_1}\right)$ *satisfies the following relation:*

$$U_m\left(A_{p_1}\right) = S\left(D_{A(2)}\left(U_m\left(0, A_{p_1}\right)\right), \left\{D_{A(i+2)}\left(R\left(J(i), p_{i+1}, A_{p_1}\right)\right),\ i = 0, \ldots, N-1\right\}\right),$$

where $J(0) = 1,\ J(n) = \prod_{r=1}^{n} p_r,\ n = 1,..., N$ and $A(i) = \prod_{r=i}^{N} p_r,\ i = 1,..., N,\ A(N+1) = 1.$

Proof By the recursion equation of Definition 5 we have:

for $n = 1$: $U_m\left(1, A_{p_1}\right) = S\left(U_m\left(0, A_{p_1}\right), R\left(1, p_1, A_{p_1}\right)\right),$

for $n = 2$: $U_m\left(2, A_{p_1}\right) = S\left(D_{p_2}\left(U_m\left(1, A_{p_1}\right)\right), R\left(J(1), p_2, A_{p_1}\right)\right)$

$$U_m\left(2, A_{p_1}\right) = S\left(D_{p_2}\left(S\left(U_m\left(0, A_{p_1}\right), R\left(1, p_1, A_{p_1}\right)\right)\right), R\left(J(1), p_2, A_{p_1}\right)\right),$$

so by definitions 1 and 2 we have:

$$U_m\left(2, A_{p_1}\right) = S\left(D_{p_2}\left(U_m\left(0, A_{p_1}\right)\right), D_{p_2}\left(R\left(1, p_1, A_{p_1}\right), R\left(J(1), p_2, A_{p_1}\right)\right)\right).$$

From definition 1, it easy to see that $D_pD_q = D_{pq}$.

Inductively, we have for $n = N$:

$$U_m\left(A_{p_1}\right) = S\left(D_{p_2\ldots p_N}\left(U_m\left(0, A_{p_1}\right)\right),\ D_{p_2\ldots p_N}\left(R\left(1, p_1, A_{p_1}\right)\right),\ D_{p_3\ldots p_N}\left(R\left(J(1), p_2, A_{p_1}\right)\right), \ldots,\right.$$

$$D_{p_N}\left(R\left(J(N-2), p_{N-1}, A_{p_1}\right)\right),\ R\left(J(N-1), p_N, A_{p_1}\right)\Big) =$$

$$= S\left(D_{A(2)}\left(U_m\left(0, A_{p_1}\right)\right), \left\{D_{A(i+2)}\left(R\left(J(i), p_{i+1}, A_{p_1}\right)\right),\ i = 0, \ldots, N-1\right\}\right). \square$$

Theorem 2 *Let* $\left(U_m\left(A_{p_1}\right)\right)_s = \left\{\left(U_m\left(A_{p_1}\right)\right)_{s,1}, \ldots, \left(U_m\left(A_{p_1}\right)\right)_{s,m}\right\}$ *be the* s *row of the matrix* $U_m\left(A_{p_1}\right)$, *where* J(i)+1 ≤ s ≤ J(i+1) *and* i = 0,…,N-1. *If* s = kJ(i) + r, k = 1,…,p_{i+1} –1, r = 1,…,J(i), *then:*

$$\left(U_m\left(A_{p_1}\right)\right)_{s,n} = \begin{cases} a_{\left\lceil \frac{np_1}{m} \right\rceil}, & \text{if } (r-1)A(i+1) + (k-1)A(i+2) + 1 \le n \le (r-1)A(i+1) + kA(i+2) \\ 0, & \text{otherwise} \end{cases},$$

where *J(i), A(i+1)* as in (1).

Proof From Theorem 1 we have:

$$U_m\left(A_{p_1}\right) = S\left(D_{A(2)}\left(U_m\left(0, A_{p_1}\right)\right), \left\{D_{A(i+2)}\left(R\left(J(i), p_{i+1}, A_{p_1}\right)\right),\ i = 0, \ldots, N-1\right\}\right),$$

$$\left(U_m\left(A_{p_1}\right)\right)_{s,n} = \left(U_m\left(A_{p_1}\right)\right)_{kJ(i)+r,n} = \left(D_{A(i+2)}\left(R\left(J(i), p_{i+1}, A_{p_1}\right)\right)\right)_{\tau,n},\ \tau = 1, \ldots, J(i)(p_{i+1}\text{-}1).$$

From definition 4 we have that:

$$\left(R\left(J(i), p_{i+1}, A_{p_1}\right)\right)_{\tau,z} = \begin{cases} a_{\left\lceil \frac{zp_1}{m} \right\rceil}, & \text{if } z = p_{i+1}Mod\left(\tau - 1, J(i)\right) + \left\lceil \frac{\tau}{J(i)} \right\rceil,\ \tau = 1, \ldots, J(i)(p_{i+1} - 1),\ z = 1, \ldots, J(i+1) \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} a_{\left\lceil \frac{zp_1}{m} \right\rceil}, & \text{if } z = p_{i+1}(r-1)+k \\ 0, & \text{otherwise} \end{cases},$$

because, $\tau = (k-1)p_{i+1} + r$, where $k = 1,\ldots, p_{i+1}\text{-}1,\ r = 1,...,J(i),$

so, $p_{i+1}Mod(\tau - 1, J(i)) + \lceil \tau / J(i) \rceil = (r-1)p_{i+1} + k.$

Consequently, we have: $\left(D_{A(i+2)}\left(R\left(J(i), p_{i+1}, A_{p_1}\right)\right)\right)_{\tau,n} =$

$$= \begin{cases} a_{\left\lceil \frac{zp_1}{m} \right\rceil}, & \text{if } z = p_{i+1}(r-1)+k \text{ and } (z-1)A(i+2)+1 \le n \le zA(i+2) \\ 0, & \text{otherwise} \end{cases}, \text{ (see Definition 1)}$$

where $r = 1,...,J(i),\ k = 1,\ldots, p_{i+1}\text{-}1,\ z = 1,...,J(i+1),\ n = 1,...,m.$

So, $\left(D_{A(i+2)}\left(R\left(J(i), p_{i+1}, A_{p_{i+1}}\right)\right)\right)_{\tau,n} =$

$$= \begin{cases} a_{\left\lceil \frac{np_1}{m} \right\rceil}, & \text{if } \left(p_{i+1}(r-1)+k-1\right)A(i+2)+1 \le n \le \left(p_{i+1}(r-1)+k\right)A(i+2) \\ 0, & \textit{otherwise} \end{cases},$$

$$= \begin{cases} a_{\left\lceil \frac{zp_1}{m} \right\rceil}, & \text{if } (r-1)A(i+1)+(k-1)A(i+2)+1 \le n \le (r-1)A(i+1)+kA(i+2) \\ 0, & \textit{otherwise} \end{cases},$$

where $r = 1,...,J(i),\ k = 1,\ldots, p_{i+1}\text{-}1, n = 1,...,m$. □

In [1] it is proved that the matrices $U_m(A_{p_1})$ are invertible and the inverse matrices are constructed by a recursion equation. Here we demonstrate the formula for their construction.

Proposition 1 Let e_l, $l=1,...,m$ be a row of $\left(\left(U_m\left(A_{p_1}\right)\right)^{-1}\right)^T$, such that $J(i)+1 \le l \le J(i+1)$, where $i = 0,...,N\text{-}1$. Take $l = k\,J(i) + r$, where $k = 1,...,p_{i+1} - 1$, $r = 1,...,J(i)$, then we have:

$$e_{l,n} = e_{kJ(i)+r,n} = \begin{cases} 1/a_{\left\lceil \frac{np_1}{m} \right\rceil}, & \textit{whenever } n = (r-1)A(i+1)+kA(i+2) \\ -1/a_{\left\lceil \frac{np_1}{m} \right\rceil}, & \textit{whenever } n = rA(i+1) \\ 0, & \textit{for all other } n\textit{'s} \end{cases},$$

where the sequences $J(n)$ και $A(i)$ have been defined in (1).

Proof See [1]. □

Example: If $m = 12$ and $p_1 = 3$, then $p_2 = p_3 = 2$, so:

$$\left(U_{12}\left(1,A_3\right)\right)^{-1}=\begin{pmatrix}0 & 1/a_1 & 0\\ 0 & 0 & 1/a_2\\ 1/a_3 & -1/a_3 & -1/a_3\end{pmatrix},\ \left(U_{12}\left(2,A_3\right)\right)^{-1}=\begin{pmatrix}0 & 0 & 0 & 1/a_1 & 0 & 0\\ 0 & 1/a_1 & 0 & -1/a_1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1/a_2 & 0\\ 0 & 0 & 1/a_2 & 0 & -1/a_2 & 0\\ 0 & 0 & 0 & 0 & 0 & 1/a_3\\ 1/a_3 & -1/a_3 & -1/a_3 & 0 & 0 & -1/a_3\end{pmatrix},$$

$$\left(U_{12}\left(3,A_3\right)\right)^{-1}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1/a_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1/a_1 & 0 & 0 & -1/a_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_1 & 0 & 0 & 0 & 0\\ 0 & 1/a_1 & 0 & -1/a_1 & 0 & 0 & 0 & -1/a_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1/a_2 & 0 & 0 & 0 & -1/a_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_2 & 0 & 0\\ 0 & 0 & 1/a_2 & 0 & -1/a_2 & 0 & 0 & 0 & 0 & -1/a_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_3 & 0\\ 0 & 0 & 0 & 0 & 0 & 1/a_3 & 0 & 0 & 0 & 0 & -1/a_3 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_3\\ 1/a_3 & -1/a_3 & -1/a_3 & 0 & 0 & -1/a_3 & 0 & 0 & 0 & 0 & 0 & -1/a_3\end{pmatrix}.$$

3. THE LINEAR TRANSFORM OF THE SPARSE MATRICES WITH WEIGHTS

Let V_m be the space consisting of all real valued sequences of length m, where m satisfies (1). i.e. $m = l\,p_1$, where $l \geq 1$ and $m = p_1 p_2 \ldots p_N$, where $p_2 \geq p_3 \geq \ldots \geq p_N$ are prime factors of m/p_1. Then, there exists a unique sequence of scalaras $\boldsymbol{b} = \{b_n: n=1,\ldots,m\}$, such that any element $\boldsymbol{t} = \{t(n): n=1,\ldots,m\} \in V_m$ can be written as:

$$\boldsymbol{t} = \sum_{i=1}^{m} b_i \left(U_m\left(A_{p_1}\right)\right)_i,$$

where $\left(U_m\left(A_{p_1}\right)\right)_i$ is the i row of the matrix $U_m\left(A_{p_1}\right)$. In addition, there holds $b_i = \left\langle t(.), \left(U_m\left(A_{p_1}\right)\right)^{-1}_{.,i} \right\rangle$, where <.,.> is the inner product of V_m.

Deifinition 6 We say that the sequence $\{V_0 \subset ... \subset V_N = V_m\}$ of subspaces of V_m, such that the set $\{\left(U_m\left(A_{p_1}\right)\right)_r : r = 1,...,J(i)\}$ *spans* V_i is a weighted multiscale analysis of the space V_m.

Example: Let $m = 12$ and $p_1 = 3$, so $p_2 = 2$, $p_2 = 2$, then $N = 3$. the subspaces V_i, $i = 0,...,3$ are the following:

If $i = 0$ we have $J(0) = 1$, so $V_0 = U_1^m$.

If $i = 1$ we have $J(1) = 3$, so: $V_1 = \{U_1^m, U_2^m, U_3^m\}$.

If $i = 2$ we have $J(2) = 6$, so: $V_2 = \{U_1^m, \ldots, U_6^m\}$.

If $i = 3$ we have $J(3) = 12$, so: $V_3 = \{U_1^m, \ldots, U_{12}^m\} = V_{12}$.

In other words, the transform induced by the matrices $U_m(A_{p_1})$ is suitable for revealing information at different scales.

Example: For $m=12$ and a sequence of real numbers $\{t_1,...,t_{12}\}$ we have:

$$\begin{pmatrix}
a_1 & a_1 & a_1 & a_1 & a_2 & a_2 & a_2 & a_2 & a_3 & a_3 & a_3 & a_3 \\
a_1 & a_1 & a_1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_2 & a_2 & a_2 & a_2 & 0 & 0 & 0 & 0 \\
a_1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_2 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & a_3 & 0 & 0 \\
a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0
\end{pmatrix} \cdot \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{pmatrix} = \begin{pmatrix}
a_1(t_1+t_2+t_3+t_4)+a_2(t_5+t_6+t_7+t_8)+a_3(t_9+t_{10}+t_{11}+t_{12}) \\
a_1(t_1+t_2+t_3+t_4) \\
a_2(t_5+t_6+t_7+t_8) \\
a_1(t_1+t_2) \\
a_2(t_5+t_6) \\
a_3(t_9+t_{10}) \\
a_1t_1 \\
a_1t_3 \\
a_2t_5 \\
a_2t_7 \\
a_3t_9 \\
a_3t_{11}
\end{pmatrix}.$$

The coding of local information is achieved in the following sense. The first element of the transform is all the information (it is the weighted mean). The 2nd and the 3rd element are the information compressed in quadruplets, the 4th, 5th and 6th elements are the information compressed in dyads and the last 6 elements are the information compressed in units.

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