DISPROPORTIONALITY OF ALLOCATION AS DETERMINED BY BOUNDARY CONDITIONS

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Abstract

According to the principle of degressively proportional allocation, larger units must limit their share in common good to the advantage of smaller units. This is only a framework principle leaving out many solutions. One way to achieve more accuracy is to define the so-called boundary conditions: to determine the minimum and maximum values of the shared good to be allocated to units participating in the division. Obviously, these values cause a particular distribution to deviate from the classical proportional allocation. This paper explores to what extent boundary conditions determine the minimum span of the deviation.

Key words: *boundary conditions, degressively proportional allocation, elections, European Parliament, fair division*

1. INTRODUCTION

Shaped by centuries in the European cultural circle, the principle of distribution of benefits and burdens derives from Aristotle's principle of proportionality. According to it, each unit participating in the division receives a share proportional to its value among all other units. The value of a unit may be determined in different ways. In a classic example of distribution of seats in collegial bodies it is the most commonly the number of people living in the constituency. It may happen so that, especially in the case of integer distributions, some individuals with low values will be deprived of any share in the benefits and burdens. Their participation expressed with a real number may be so small that with certain types of roundings will be set at zero. Such a situation is not always acceptable. There may occur circumstances in which evident, expressed with a non-zero share, consession of all individual units to the distribution is valuable. One can also imagine situations that require a minimum share for each unit. Then there exists a need to modify the generally accepted principles of proportional distribution.

Existence of the circumstances described above resulted in formation of the principle of degressively proportional distribution. It was entered as applicable law in the Lisbon Treaty in response to the need to ensure a certain minimum representation in the European Parliament for each Member State. Assuming a reasonable size of the parliamentary chamber, the rule of proportional distribution would deprive eight of the smallest of the European Union countries of suitable representation that the Treaty sets at six seats. Exactly the Treaty of Lisbon states that *"The European Parliament shall be composed of representatives of the Union's citizens. They shall not exceed seven hundred and fifty in number, plus the President. Representation of citizens shall be degressively proportional, with a minimum threshold of six members per Member State. No Member State shall be allocated more than ninety-six seats."* (The Treaty of Lisbon 2010).

What exactly the authors of the concept of understand under the notion of degressive proportionality has been clarified in the report of Lamassoure and Severin (Lamassoure, Severin 2007), where one can find that digressive proportionality means that "the larger the population of a country, the more inhabitants are represented by each of its Members of the European Parliament". This makes it possible to precisely define the princip in a mathematical way: positive sequence $s_1, s_2, \ldots, s_n$ is degressively proportional with respect to $0 < p_1 \le p_2 \le \ldots \le p_n$ if and only if $s_1 \le s_2 \le \ldots \le s_n$ and $\frac{p_1}{s_1} \le \frac{p_2}{s_2} \le \ldots \le \frac{p_n}{s_n}$. Sequence $S = (s_1, s_2, \ldots, s_n)$ is, of course, in our case a sequence of terms defining the total number of allocated seats, while the sequence $P = (p_1, p_2, \ldots, p_n)$ is a seguence expressing the values of individual units, which in this case is the population of each Member State of the European Union. The overall situation of the sequence *S* can have real values corresponding to allocations of individual units.

In the same language the proportional allocation rule can be written as follows: positive sequence $q_1, q_2, ..., q_n$ is proportional with respect to $0 < p_1 \leq p_2 \leq ... \leq p_n$ if and only if $\frac{p_1}{q_1} = \frac{p_2}{q_2} = ... = \frac{p_n}{q_n}$. Sequence $Q = (q_1, q_2, \ldots, q_n)$ with the real terms is called a sequence of quotas of proportional distribution. This sequence with a fixed number or amount of distributed good *H* is determined unambiguously and the only problem in the case of integer division is the adequate approximation. Degressively proportional sequences do not have such a property. Thus, even at a fixed *H* there are many possible ways of allocation.

In the report of Lamassoure and Severin mentioned earlier, one can find also a record stating that the minimum and maximum numbers set by the Treaty „must be fully utilised to ensure that the allocation of seats in the European Parliament reflects as closely as possible the range of populations of the Member States", in other words, specifying the boundary conditions of the distribution in such a way that $s_1 = m$ i $s_n = M$. In this case, $m = 6$ and $M = 96$ Moreover, in this sentence, the emphasis is on reflecting by the sequence *S* the structure of the population of the Member States as faithfully as possible. Therefore, this can be interpreted as a recommendation of creating degression of the distribution only by its boundary conditions. Consequently, one should seek the degressively proportional allocation most similar to the proportional distribution, satisfying the conditions $s_1 = m$ i $s_n = M$.

It turns out that it is difficult to determine what is meant by the division closest to proportional. There can be many concepts to achieve this postulate (Pukelsheim 2007; Ramirez González, Martinez Aroza, Márquez Garcia 2012; Łyko, Rot, Rudek 2012).This is connected with the problem of determining the measure of degression of a given allocation understood as a measure of the deviation from the quotas of the proportional division (Łyko 2013, Dniestrzański 2014).The rest of the work will present a way to indicate the minimum, in a certain sense, degression of the degressively division resulting from the accepted boundary conditions.

2. MEASURING DEGRESSION OF THE DEGRESSIVELY PROPORTIONAL DISTRIBUTION

Let us assume that there is given sequence $0 < p_1 \leq p_2 \leq ... \leq p_n$ and integers *H*, *m* and *M*, respectively defining the total number of shared goods and the number of goods granted to the smallest and the largest unit, which is $H = s_1 + s_2 + \ldots + s_n$ $s_1 = m$ i $s_n = M$. Then for each *i* one can specify the minimum $s_i^{\min}$ and maximum $s_i^{\max}$ number of goods that can be allocated to unit *i* in the integer manner and degressively proportional, meeting the conditions of $H = s_1 + s_2 + \ldots + s_n$ $s_1 = m$ i $s_n = M$ (Łyko, Rudek 2013). In such a case, there are disjoint subsets $N_U = \{1, 2, \ldots, k\}$, $N_{prop} = \{k + 1, k + 2, \ldots, l - 1\}$, $N_L = \{l, l + 1, \ldots, n\}$, some of them maybe empty, of a set $N = \{1, 2, \ldots, n\}$ such that for every $i \in N_U$ $q_i < s_i^{\min}$ for every $i \in N_{prop}$ $q_i \in \left\langle s_i^{\min}, s_i^{\max} \right\rangle$ and for every $i \in N_L$ $q_i > s_i^{\max}$ wherein the sequence $Q = (q_1, q_2, \ldots, q_n)$ is a sequence of quotas of the proportional distribution. A subset of N_{prop} consists of numbers of these units, for which distribution according to the rules of proportional allocation is possible. A subset N_U, is a set of the numbers of individual units that as a result of the degressively proportional distribution receive more than according to the proportional one, and a subset N_L, is a set of the numbers of units that on the contrary, receive less shared goods than according to the Aristotle's rule (Fig. 1).

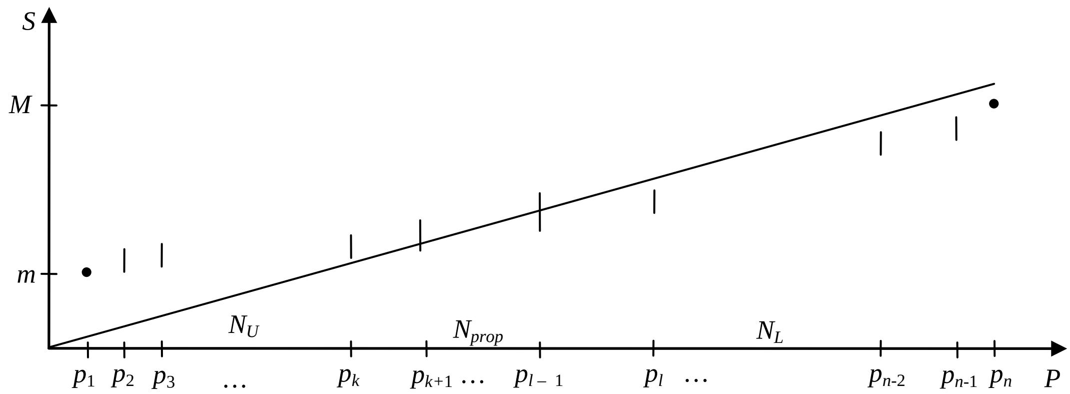

Fig. 1. Sets N_U, N_{prop} and N_L

A subset N_{prop} and hence subsets N_U and N_L can be defined differently, understanding, for example, the possibility of having proportional allocation in the sense of keeping the quota. Then, instead of the interval $\left\langle s_i^{min}, s_i^{max} \right\rangle$ one should consider interval $\left\langle s_i^{min} - 1, s_i^{max} + 1 \right\rangle$. This changes concrete results, but the concept to indicate a subset of units for which proportional allocation is possible remains unchanged.

Therefore, looking for degressively proportional sequences $R = (r_1, r_2, \ldots, r_n)$ with real values satisfying the conditions $r_1 = m$, $r_n = M$ i $r_1 + r_2 + \ldots + r_n = H$ it can be assumed that for the individuals belonging to a subset N_{prop} it will simply be a sequence of quotas of proportional distribution. This means that for $i \in N_{prop}$ $r_i = q_i$. This will split H_{prop} goods, where $H_{prop} = q_{k+1} + q_{k+2} + \ldots + q_{l-1}$. The remaining portion, namely $H_{prop} - H$ should be assigned to units belonging to subsets N_U and N_L. Let $H_U = r_1 + r_2 + \ldots + r_k$ and $H_l = r_l + r_{l+1} + \ldots + r_k$. Units belonging to a subset N_U get more of the shared goods then it results from the proportional allocation rules. Trying therefore with degressively proportional division to faithfully reflect the structure of the values of individual units, one should postulate to minimize H_U. Let $H_U^* = \inf\limits_{R \in \Re_{U(+1)}} \{r_1 + r_2 + \ldots + r_n\}$, where $\Re_{U(+1)} = (r_1, r_2, \ldots + r_{k+1})$ is the set of all degressively proportional sequences with respect to sequence $p_1 \le p_2 \le \ldots \le p_{k+1}$ such that $r_i = f_u(p_i)$ for $i \in N_U$ and $r_{k+1} = q_{k+1}$, where f_U is a linear function such that $f_U(p_1) = m$, and further there exist a linear function f_L such that $f_L(p_n) = M$ and a sequence $q_{l-1} \le f_L(p_l) \le \ldots \le f_L(p_n)$ is degressively proportional with respect to $p_{l-1} \le p_i \le \ldots \le p_n$ and $f_L(p_l) + f_L(p_{l+1}) + \ldots + f_L(p_n) = H_L^*$ where $H_L^* = H - H_U^* - H_{prop}$. Value

$$BD = \frac{H_U^* - (q_1 + q_2 + \ldots + q_k) + (q_l + q_{l+1} + \ldots + q_n) - H_L^*}{H}$$

can be regarded as a measure of the minimum deviation of the degressively proportional allocation generated by the boundary conditions $r_1 = m$, $r_n = M$.

Using the distribution of set N for the sum of disjoint subsets N_U, N_{prop} and N_L one can specify the reference model, with the set boundary conditions, namely sequence $Q^* = (q_1^*, q_2^*, \ldots, q_n^*)$ degressively proportional with respect to sequence P. It is enough to assume that the

$$q_i^* = \begin{cases} f_U(p_i) & \text{for} \quad i \in N_U \\ q_i & \text{for} \quad i \in N_{prop} \\ f_L(p_i) & \text{for} \quad i \in N_L \end{cases}$$

where f_U and f_L are linear functions such that $f_U(p_1) = m$ and $f_L(p_n) = M$, the sequences $f_U(p_1) \le f_U(p_2) \le ... \le f_U(p_k) \le q_{k+1}$ and $q_{l-1} \le f_L(p_l) \le f_L(p_{l+1}) \le \ldots \le f_L(p_n)$ are degressively proportional with respect to sequences $p_1 \le p_2 \le \ldots \le p_{k+1}$ and $p_{l-1} \le p_l \le \ldots \le p_n$ respectively and $f_U(p_1) + f_U(p_2) + \ldots + f_U(p_k) = H_U^*$ and $f_L(p_l) + f_L(p_{l+1}) + \ldots + f_L(p_n) = H_L^*$. Sequence Q^* has of course the real values and can be treated as an equivalent of quotas sequence Q for the proportional division. In the above sense, the sequence Q^* is closest to Q, ie. the sequence representing the ideal proportion, a degressively proportional sequence satisfying the fixed boundary conditions.

3. EMPIRICAL VERIFICATION IN THE CASE OF THE DISTRIBUTION OF SEATS IN THE EUROPEAN PARLIAMENT

Table 1 shows the values $s_i^{\min}$ and $s_i^{\max}$ for each Member State of the European Union. In addition, it contains the sequences Q, and Q^*, as well as the distribution of S in force from 2014 to 2019 term. A subset $N_{prop} = \{17, 18, \ldots, 24\}$, and the empirically determined value of H_U^* is 154.64. It may be noted that the actual distribution significantly deviates from the standard Q^* generated by the boundary conditions, which can be determined by summing the seats allocated to the sixteen smallest countries, namely those whose ordinal numbers are in a set N_U. In this case, the number is 188 and shows that the solution is valid for the preferred group of countries.

The value of BD minimum degression in this case is 0.133. One will notice that $H_U^* - (q_1 + q_2 + ... + q_k) + (q_l + q_{l+1} + ... + q_n) - H_L^* = |q_1^* - q_1| + |q_2^* - q_2| + ... + |q_n^* - q_n|$ ie. the distance $d_1(Q^*, Q)$ between sequences Q^* and Q in the space $\mathbb{R}^n$. It can be seen that $d_1(S, Q) = 209.80$ and $d_1(Q^*, Q) = 99.90$ and therefore, identically calculated ratio $\frac{d_1(S,\ Q)}{H}$ is 0.279 and is more than twice the size of the value of BD. Hence the conclusion that the boundary conditions established in the Treaty of Lisbon allow for the allocation seats in the European Parliament of far more closer to the proportional distribution.

Table 1. Distribution of seats in the European Parliament

i	Country	Population	$s_i^{\min}$	$s_i^{\max}$	q_i	q_i^*	s_i
1	Malta	416110	6	6	0.62	6.00	6
2	Luxembourg	524853	6	7	0.78	6.10	6
3	Cyprus	862011	6	11	1.27	6.41	6
4	Estonia	1339662	6	15	1.98	6.84	6
5	Latvia	2041763	6	15	3.02	7.49	8
6	Slovenia	2055496	6	15	3.04	7.50	8
7	Lithuania	3007758	8	16	4.45	8.37	11
8	Croatia	4398150	10	17	6.50	9.64	11
9	Ireland	4582769	10	17	6.77	9.81	11
10	Slovakia	5401267	11	17	7.98	10.56	13
11	Finland	5404322	11	17	7.99	10.56	13
12	Denmark	5580516	11	17	8.25	10.72	13
13	Bulgaria	7327224	14	19	10.83	12.32	17
14	Austria	8443018	14	20	12.48	13.34	18
15	Sweden	9482855	15	21	14.02	14.29	20
16	Hungary	9957731	15	22	14.72	14.72	21
17	Portugal	10505445	15	22	15.53	15.53	21
18	Czech Republic	10541840	15	22	15.58	15.58	21
19	Greece	11041266	15	23	16.32	16.32	21

20	Belgium	11290935	15	23	16.69	16.69	21
21	Netherlands	16730348	22	29	24.73	24.73	26
22	Romania	21355849	27	37	31.57	31.57	32
23	Poland	38538447	47	60	56.96	56.96	51
24	Spain	46196276	56	70	68.28	68.28	54
25	Italy	60820764	73	84	89.90	83.57	73
26	UK	62989550	75	85	93.11	84.85	73
27	France	65397912	77	87	96.67	86.28	74
28	Germany	81843743	96	96	120.97	96.00	96

4. CONCLUSIONS

The proposed indicator *BD* determines the minimum degression that can be achieved using piecewise linear allocation functions. This allows to determine what is the effect of boundary conditions on the deviation of degressively proportional distribution from the proportional allocation understood as a series of quotas. This makes it possible to control the boundary conditions in such a way to get a pre-planned disparity of the distribution between individuals gaining or losing as a result of rejecting the of Aristotle's rule.

Additionally, the presented construction allows for the appointment of sequence Q^*, which may play a similar role to the degressively proportional distribution, as played by the sequence of quotas for the proportional distribution. Among other things, this allows to use the method of the highest amounts by Hamilton. Key to this concept is to establish a set of N_{prop}. The approach proposed in the article is of course not the only one possible. It has the advantage of being easy from the numerical point of view, since it is not very complex to determine the value s_i^{min} and s_i^{max}. The most natural way to define a set of N_{prop} would be to define it as the maximum subset of the set N, for which there exist sequence $R = (r_1, r_2, \ldots, r_n)$ degressively proportional with respect to the sequence $P = (p_1, p_2, \ldots, p_n)$ such, that for each $i \in N_{prop}$ $r_i = q_i$ and $r_1 + r_2 + \ldots + r_n = H$, $r_1 = m$ and $r_n = M$.

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